Time-harmonic Green’s functions for anisotropic magnetoelectroelasticity

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Abstract

Two-dimensional (2-D) and three-dimensional (3-D) time-harmonic Green’s functions for linear magnetoelectroelastic solids are derived in this paper by means of Radon-transform. Displacement field and electric and magnetic potentials in a fully anisotropic magnetoelectroelastic infinite solid due to a time-harmonic point force, point charge and magnetic monopole are obtained in form of line integrals over a unit circle in 2-D case and surface integrals over a unit sphere in 3-D case. This dynamic fundamental solution is then split into the sum of regular dynamic plus singular terms. The singular terms coincide with the Green’s functions for the static problem and may be further reduced to closed form expressions. The proposed Green’s functions can be used in the corresponding boundary element method (BEM) formulation.

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1. Introduction

Smart structures applications are receiving increasing attention in recent years. Solids showing piezoelectric and piezomagnetic couplings are being widely used as ultrasonic transducers, magnetic field probes, etc. The most recent advances involve composite materials consisting of both piezoelectric and piezomagnetic phases, so that a magnetoelectric coupling effect is obtained as well. Although such coupling is not shown by any of the material phases alone, the magnetoelectric coupling of the resulting composite may be much larger that of a single phase magnetoelectric material (Alshits et al., 1995; Avellaneda and Harshe, 1994; Benveniste, 1995; Nan, 1994; Van Suchtelen, 1972).

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When numerically modeling the behavior of components made of these magnetoelectroelastic materials via boundary element methods (BEM) or boundary integral equation method (BIEM), the fundamental solutions or Green’s functions play a key role in the formulation and the resulting accuracy of the method. Green’s functions are also a key ingredient in other analytical and numerical techniques such as eigenstrain approaches or dislocation methods.

Within the framework of statics, Huang and Kuo (1997), Huang et al. (1998) and Li and Dunn (1998) used the 3-D magnetoelectroelastic Green’s functions to study the inclusion problems and the effective behavior of magnetoelectroelastic composite materials. Liu et al. (2001) derived the Green’s functions for a 2-D anisotropic magnetoelectroelastic solid containing an elliptical cavity based on the extended Stroh’s formalism combined with the conformal mapping technique and the Laurent series expansion. Pan (2002) obtained the 3-D Green’s functions in anisotropic magnetoelectroelastic full space, half space, and bimaterials based on the extended Stroh’s formalism by applying the 2-D Fourier transforms. Wang and Shen (2002) obtained the general solution of 3-D problems in magnetoelectroelastic media in terms of five potential functions, and subsequently applied it to obtain the fundamental solution for a generalized dislocation and to derive Green’s functions for a semi-infinite magnetoelectroelastic solid. Ding and Jiang (2004) presented a boundary integral formulation for 2-D problems in magnetoelectroelastic media and derived the corresponding fundamental solution in closed form. Qin (2004) derived Green’s functions for magnetoelectroelastic medium with an arbitrarily oriented half-plane or bimaterial interface by means of an extended Stroh’s formalism and coordinate transform technique. Jiang and Pan (2004) obtained the 2-D Green’s function in terms of the Stroh formalism in an exact closed form for magnetoelectroelastic full-, half-, and bimaterial-planes with general anisotropy. Hou et al. (2005) presented 3-D Green’s functions of infinite, two-phase and semi-infinite transversely isotropic magnetoelectroelastic media in terms of mono-harmonic displacement functions for all cases of distinct eigenvalues and multiple eigenvalues. Ding et al. (2005) obtained Green’s functions for two-phase transversely isotropic magnetoelectroelastic media in the case of distinct eigenvalues, including 2-D Green’s functions of a two-phase infinite plane and half-plane as well as 3-D counterparts. Guan and He (2005) and Guan and He (2006) derived the general solution of governing equations to 2-D and 3-D problems in a transversely isotropic magnetoelectroelastic media in terms of potential functions. These general solutions are then applied to obtain fundamental solutions in some cases of loading for half-plane and half-space.

However, unlike in statics, relatively little work has been done regarding dynamic Green’s functions of magnetoelectroelastic materials. Ren and Liu (2004) presented a time-harmonic dynamic fundamental solution for transversely isotropic magnetoelectroelastic media under anti-plane deformation. Chen et al. (2006) derived the explicit expressions for the dynamic potentials of an inclusion embedded in a ‘quasi-plane’ magnetoelectroelastic medium of transversely isotropic symmetry and for the dynamic Green’s functions of such medium both in the time domain and in the frequency domain.

Although the 2-D and 3-D dynamic fundamental solutions for anisotropic elastic and piezoelectric media have been studied in detail (see for instance Denda et al., 2004; Wang and Achenbach, 1994; Wang and Zhang, 2005), dynamic Green’s functions for 2-D and 3-D fully anisotropic magnetoelectroelastic materials are still unavailable in the literature to the authors’ knowledge. In this paper, the Radon-transform technique presented by Wang and Achenbach (1994), Wang and Achenbach (1995) for anisotropic elastic solids is extended to derive the dynamic Green’s functions for 2-D and 3-D anisotropic magnetoelectroelastic media subjected to time-harmonic loading conditions. This same technique was successfully applied by Denda et al. (2004) and Wang and Zhang (2005) to derive the fundamental solutions for dynamic piezoelectricity. As for anisotropic elastic and piezoelectric solids, the dynamic Green’s functions are not obtained in closed form, but may be expressed as surface integrals over a unit sphere in 3-D case and as line integrals over a unit circle in 2-D case. The dynamic Green’s functions derived in this way can be further decomposed into a singular part and a regular part. The singular part corresponds to the static magnetoelectroelastic Green’s functions, whilst the regular part represents the contribution of the inertial terms in the equations of motion. When the piezomagnetic and electromagnetic constants vanish, our results reduce to existing solutions of the piezoelectric solids (Wang and Zhang, 2005).

The obtained time-harmonic fundamental solutions may be used in combination with the BEM to analyse dynamic problems in linear magnetoelectroelastic solids, since they exhibit a simple mathematical structure for an easy numerical implementation, as previous works illustrate for both the anisotropic (Sáez and Domínguez, 1999; Wang et al., 1996) and piezoelectric cases (Sáez et al., 2006).
2. Basic equations of linear magnetoelectroelasticity

Due to the coupling of elastic, electric and magnetic behaviors, the constitutive equations for a homogeneous, linear and fully anisotropic magnetoelectroelastic solid are given by

\[
\begin{align*}
\sigma_{ij} &= C_{ijkl} \varepsilon_{kl} - d_{ij} E_l - h_{ij} H_l \\
D_i &= e_{ikl} \varepsilon_{kl} + \varepsilon_{li} E_l + \beta_{ij} H_l \\
B_i &= h_{ikl} \varepsilon_{kl} + \beta_{jl} E_l + \gamma_{ij} H_l
\end{align*}
\]

where the summation rule over double indices is applied; \( \sigma_{ij} \), \( D_i \) and \( B_i \) are the mechanical stresses, the electric displacements and the magnetic inductions, respectively; \( C_{ijkl} \), \( \varepsilon_{ij} \) and \( \gamma_{ij} \) denote elastic stiffness tensor, dielectric permitivities and magnetic permeabilities, respectively; \( e_{ikl} \), \( h_{ij} \) and \( \beta_{ij} \) are the piezoelectric, piezomagnetic and magnetoelectric coupling coefficients, respectively; and \( \varepsilon_{ij} \), \( E_i \) and \( H_i \) are the mechanical strains, the electric field and the magnetic field, respectively

\[
\begin{align*}
\varepsilon_{ij} &= \frac{1}{2} \left( u_{ij,j} + u_{ji,j} \right) \\
E_i &= -\phi_i \\
H_i &= -\varphi_i
\end{align*}
\]

where \( u_i \) are the elastic displacements, \( \phi \) is the electric potential and \( \varphi \) is the magnetic potential.

The material constants tensors show the following symmetry conditions

\[
C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}; \quad e_{kj} = e_{jk}; \quad h_{kl} = h_{lk}; \quad \varepsilon_{kl} = \varepsilon_{lk}; \quad \beta_{kl} = \beta_{lk}; \quad \gamma_{kl} = \gamma_{lk}
\]

Moreover, the elastic stiffnesses, dielectric permitivities and magnetic permeabilities are positive definite

\[
C_{ijkl} e_{kl} e_{ij} > 0; \quad \varepsilon_{kl} E_k E_i > 0; \quad \gamma_{kl} H_k H_i > 0
\]

for real non-zero \( \varepsilon_{ij} \), \( E_k \) and \( H_k \) (Soh and Liu, 2005).

The dynamic equilibrium equations and the Maxwell equations under the assumption of quasi-static electric and magnetic fields are given by

\[
\begin{align*}
\sigma_{ij,j} &= -b_i + \rho \frac{\varepsilon^2}{c^2} u_i \\
D_{i,i} &= f_e \\
B_{i,i} &= f_m
\end{align*}
\]

where \( \rho \) is the mass density, \( b_i \) are the body forces and \( f_e \) and \( f_m \) are the electric charge density and the electric current density, respectively.

The linear magnetoelectroelastic problem may be formulated in an elastic-like fashion by considering a generalized displacement vector extended with the electric potential and the magnetic potential as

\[
u_I = \begin{cases} 
  u_i & I = 1, 2, 3 \\
  \phi & I = 4 \\
  \varphi & I = 5 
\end{cases}
\]

and a generalized stress tensor extended with the electric displacements and the magnetic inductions as

\[
\sigma_{IJ} = \begin{cases} 
  \sigma_{ij} & J = 1, 2, 3 \\
  D_i & J = 4 \\
  B_i & J = 5 
\end{cases}
\]

In this way, the generalized equations of motion given in Eqs. (9)–(11) can be rewritten in terms of the generalized displacements as
\[ C_{ijkl}u_{K,il} = -F_J + \rho \delta^*_J \frac{\partial^2 U_K}{\partial t^2} \]  

(14)

where \( \delta^*_J \) is the generalized Kronecker delta defined by

\[
\delta^*_J = \begin{cases} 
\delta_{jk} & J, K = 1, 2, 3 \\
0 & \text{otherwise} 
\end{cases}
\]  

(15)

\( F_J \) is the corresponding generalized body forces vector

\[
F_J = \begin{cases} 
h_J & J = 1, 2, 3 \\
-f_e & J = 4 \\
-f_m & J = 5 
\end{cases}
\]  

(16)

and the elastic, electric and magnetic moduli have been grouped together into a generalized elasticity tensor as

\[
C_{ijkl} = \begin{cases} 
C_{ijkl} & J, K = 1, 2, 3 \\
e_{ij} & J = 1, 2, 3; \ K = 4 \\
h_{ij} & J = 1, 2, 3; \ K = 5 \\
e_{ikl} & J = 4; \ K = 1, 2, 3 \\
h_{ikl} & J = 5; \ K = 1, 2, 3 \\
-\beta_{kl} & J = 5; \ K = 4 \\
-\gamma_{il} & J, K = 5 
\end{cases}
\]  

(17)

so that the constitutive Eqs. (1)–(3) can be rewritten together as

\[
\sigma_{ij} = C_{ijkl}u_{K,ij}
\]  

(18)

where the lowercase (elastic) and uppercase (extended) subscripts take values 1, 2, 3 and 1, 2, 3 (elastic), 4 (electric), 5 (magnetic), respectively.

3. Time-harmonic Green’s functions

3.1. 3-D time-harmonic Green’s functions

The Green’s functions are defined as the response of an infinite homogeneous 3-D linear magnetoelectroelastic solid due to the application of a time-harmonic point force (in the generalized or extended sense) at the origin in the \( x_M \)-direction

\[
F_J(x,t) = \delta_{JM} \delta(x)e^{-i\omega t}
\]  

(19)

where \( \omega \) is the angular frequency of excitation and \( \delta(x) \) is the Dirac delta function. The resulting generalized displacement field is in the steady state of harmonic motion

\[
u_K(x,t) = U_{KM}(x, x)e^{-i\omega t}
\]  

(20)

Substitution into the generalized equations of motion (14) leads to

\[ C_{ijkl}U_{K,il}(x, \omega) + \rho \omega^2 \delta^*_J U_{KM}(x, \omega) = -\delta_{JM} \delta(x) \]  

(21)

To solve this set of Eq. (21) the Radon-transform technique (see Appendix A) will be used. This technique has been successfully applied by Wang and Achenbach (1994) and Wang and Achenbach (1995) to derive time-harmonic Green’s functions for linear anisotropic elastic solids, and by Denda et al. (2004) and Wang and Zhang (2005) for linear fully anisotropic piezoelectric solids. In this way, application of Radon-transform to (21) will reduce the 3-D equations of motion to their one-dimensional (1-D) counterparts. Once the result-
ing 1-D wave equations are solved, the solutions of the 3-D equations of motion will follow by a simple application of the inverse Radon-transform to yield the Green’s functions in the form of surface integrals over a unit sphere.

By applying the Radon-transform (see Appendix A) to Eq. (21) yields

$$\Gamma_{JK}\partial_s^2\tilde{U}_{KM}(s,\omega) + \rho \omega^2 \delta_{jk} \tilde{U}_{KM}(s,\omega) = -\delta_{jm}\delta(s)$$

(22)

where $s$ is the parameter of the Radon-transform, defined by $s = n \cdot x$ with $n$ being a unit normal vector, and $\Gamma_{JK}$ is the generalized Christoffel tensor given by

$$\Gamma_{JK} = C_{JKln}m_l$$

(23)

The solution to Eq. (22) may be obtained as the superposition of the following three cases.

3.1.1. Generalized displacements due to the application of a mechanical point load

When a mechanical point load is applied at the origin $x = 0$ in the $x_m$-direction, the elastic displacements $U_{km}$ at a point $x$ in the $x_k$-direction, the electric potential $U_{4m}$ at $x$ and the magnetic potential $U_{5m}$ at $x$ are obtained from the following set of equations

$$\Gamma_{jk}\partial_s^2\tilde{U}_{km} + \Gamma_{jl}\partial_s^2\tilde{U}_{4m} + \Gamma_{jm}\partial_s^2\tilde{U}_{5m} + \rho \omega^2 \delta_{jk} \tilde{U}_{km} = -\delta_{jm}\delta(s)$$

(24)

$$\Gamma_{4k}\partial_s^2\tilde{U}_{km} + \Gamma_{4k}\partial_s^2\tilde{U}_{4m} + \Gamma_{4s}\partial_s^2\tilde{U}_{5m} = 0$$

(25)

$$\Gamma_{5k}\partial_s^2\tilde{U}_{km} + \Gamma_{5k}\partial_s^2\tilde{U}_{4m} + \Gamma_{5s}\partial_s^2\tilde{U}_{5m} = 0$$

(26)

3.1.2. Generalized displacements due to the application of a point charge

When an electrical point charge is applied at the origin $x = 0$ in the $x_k$-direction ($U_{4k}$), the electric potential at $x$ ($U_{44}$) and the magnetic potential at $x$ ($U_{54}$) are obtained from the following set of equations

$$\Gamma_{jk}\partial_s^2\tilde{U}_{4k} + \Gamma_{jk}\partial_s^2\tilde{U}_{44} + \Gamma_{js}\partial_s^2\tilde{U}_{54} + \rho \omega^2 \delta_{jk} \tilde{U}_{44} = 0$$

(27)

$$\Gamma_{4k}\partial_s^2\tilde{U}_{4k} + \Gamma_{4k}\partial_s^2\tilde{U}_{44} + \Gamma_{4s}\partial_s^2\tilde{U}_{54} = -\delta(s)$$

(28)

$$\Gamma_{5k}\partial_s^2\tilde{U}_{4k} + \Gamma_{5k}\partial_s^2\tilde{U}_{44} + \Gamma_{5s}\partial_s^2\tilde{U}_{54} = 0$$

(29)

3.1.3. Generalized displacements due to the application of a magnetic monopole

When a magnetic monopole is applied at the origin $x = 0$ in the $x_k$-direction ($U_{5k}$), the electric potential at $x$ ($U_{45}$) and the magnetic potential at $x$ ($U_{55}$) are obtained from the following set of equations

$$\Gamma_{jk}\partial_s^2\tilde{U}_{5k} + \Gamma_{jk}\partial_s^2\tilde{U}_{45} + \Gamma_{js}\partial_s^2\tilde{U}_{55} + \rho \omega^2 \delta_{jk} \tilde{U}_{55} = 0$$

(30)

$$\Gamma_{4k}\partial_s^2\tilde{U}_{5k} + \Gamma_{4k}\partial_s^2\tilde{U}_{45} + \Gamma_{4s}\partial_s^2\tilde{U}_{55} = 0$$

(31)

$$\Gamma_{5k}\partial_s^2\tilde{U}_{5k} + \Gamma_{5k}\partial_s^2\tilde{U}_{45} + \Gamma_{5s}\partial_s^2\tilde{U}_{55} = -\delta(s)$$

(32)

The solution to Problem $A$ (mechanical load) is first considered. From Eqs. (25) and (26) $\tilde{U}_{4m}$ and $\tilde{U}_{5m}$ may be expressed as functions of $\tilde{U}_{km}$ yielding

$$\partial_s^2\tilde{U}_{4m} = \frac{\Gamma_{4k}\tilde{U}_{55} - \Gamma_{4s}\tilde{U}_{55}}{\Gamma_{4s}\tilde{U}_{54} - \Gamma_{4k}\tilde{U}_{54}} \partial_s^2\tilde{U}_{km} = \alpha_{44}\partial_s^2\tilde{U}_{km}$$

(33)

$$\partial_s^2\tilde{U}_{5m} = \frac{\Gamma_{5k}\tilde{U}_{55} - \Gamma_{5s}\tilde{U}_{55}}{\Gamma_{5s}\tilde{U}_{54} - \Gamma_{5k}\tilde{U}_{54}} \partial_s^2\tilde{U}_{km} = \alpha_{55}\partial_s^2\tilde{U}_{km}$$

(34)

Substitution of these relations in Eq. (24) yields

$$\{Z_{jk}\partial_s^2 + \rho \omega^2 \delta_{jk}\} \tilde{U}_{km} = -\delta_{jm}\delta(s)$$

(35)

where
\[ Z_{jk} = \Gamma_{jk} + x_k^4 \Gamma_{jk} + x_j^4 \Gamma_{j5} \]  

(36)

By virtue of Eqs. (7) and (8), \( Z_{jk} \) is symmetric and positive definite, so that its eigenvalues are real-valued and positive.

Denoting the eigenvalues for \( Z_{jk} \) as \( \lambda_q = \rho c_q^2 \) with \( c_q \) being the phase velocities, they are obtained as the roots of the following characteristic equations

\[ \det(Z_{jk} - \rho c_q^2 \delta_{jk}) = 0 \]  

(37)

Calling \( V_{jq} \) to the \( q \)th eigenvector of \( Z_{jk} \)

\[ Z_{jk} V_{kj} = \lambda_q V_{jq} \quad \text{(no sum on} q) \]  

(38)

it holds that

\[ V_{jq} V_{jq} = \delta_{pq} ; \quad V_{iq} V_{jq} = \delta_{ij} \]  

(39)

so that these eigenvectors may be taken as orthonormal bases. \( \hat{U}_{km} \) can then be expressed in the new bases by

\[ \hat{U}_{km} = V_{kh} \hat{U}_{km} \Leftrightarrow \hat{U}_{km} = V_{kh} \hat{U}_{km} \]  

(40)

Substituting this bases transformation into Eq. (24) and premultiplying the resulting equation by \( V_{jq} \)

\[ \{ V_{jq} Z_{jk} V_{kh} \hat{U}_{km} + \rho \omega^2 V_{jq} \delta_{jk} V_{kh} \} \hat{U}_{km} = -\delta_{jm} V_{jq} \delta(s) \]  

(41)

Taking into account relations (38) and (39), Eq. (41) may be further reduced to the following 1-D wave equation (for each fixed \( q \) and \( m \))

\[ \{ \lambda_q \delta_{2} + \rho \omega^2 \} \hat{U}_{qm} = -V_{mq} \delta(s) \]  

(42)

whose well-known solution is given by

\[ \hat{U}_{qm} = \frac{i V_{mq}}{2 \rho c_q^2 k_q} e^{ik_q x} \]  

(43)

where \( k_q \) is the wave number

\[ k_q = \frac{\omega}{c_q} \]  

(44)

The inverse bases transformation Eq. (40) leads to

\[ V_{km} = \frac{i V_{kq} V_{mq}}{2 \rho c_q^2 k_q} e^{ik_q x} \]  

(45)

Computation of \( \hat{U}_{km} \) according to Eq. (45) would require solving the complete eigenvalue problem defined by (37). This can be avoided by following the alternative approach proposed by Wang and Achenbach (1995) for anisotropic solids. In this way, the term \( V_{kq} V_{mq} \) in (45) may be computed as

\[ V_{kq} V_{mq} = \frac{E_{km}^3}{E_{pp}} \]  

(46)

where

\[ E_{km} = \text{adj} \{ Z_{km} - \rho c_q^2 \delta_{km} \} \]  

(47)

More simplified expressions for the cases in which the three phase velocities \( c_q \) are not distinct may be obtained following Wang and Achenbach (1994). Thus, Eq. (45) can be expressed as

\[ \hat{U}_{km} = \frac{i E_{km}}{2 \rho c_q^2 k_q E_{pp}} e^{ik_q x} \]  

(48)

Subsequent application of the inverse Radon-transform leads to the \( U_{km} \) terms of the Green’s functions.
where the domain of integration is defined by the surface of a unit sphere \(|n| = 1\).

Similarly, substitution of Eq. (48) into (33) and (34) yields \(\hat{U}_{4m}\) and \(\hat{U}_{5m}\), so that the subsequent application of the inverse Radon-transform leads to

\[
\begin{align*}
U_{4m} &= \frac{1}{16\pi^2} \int_{|n|=1} \frac{\alpha_{4m} E^q_{lm}}{\rho c^2_q E^q_{pp}} \left\{ 2\delta(n \cdot x) + ik_q \right\} e^{ik_q |n| x} \, dS(n) \\
U_{5m} &= \frac{1}{16\pi^2} \int_{|n|=1} \frac{\alpha_{5m} E^q_{lm}}{\rho c^2_q E^q_{pp}} \left\{ 2\delta(n \cdot x) + ik_q \right\} e^{ik_q |n| x} \, dS(n)
\end{align*}
\]

where \(\alpha_{4m}\) and \(\alpha_{5m}\) have been previously defined in (33) and (34).

Following essentially the same procedure, Green’s functions for Problem B (electric load) and Problem C (magnetic load) may be obtained. Therefore for Problem B \(\partial_z^2 \hat{U}_{44}\) and \(\partial_y^2 \hat{U}_{54}\) may be expressed as functions of \(\partial_x^2 \hat{U}_{44}\) from Eqs. (28) and (29) yielding

\[
\begin{align*}
\partial_x^2 \hat{U}_{44} &= \alpha_{44} \partial_x^2 \hat{U}_{44} + \frac{\Gamma_{55}}{\alpha} \delta(s) \\
\partial_y^2 \hat{U}_{54} &= \alpha_{54} \partial_y^2 \hat{U}_{54} - \frac{\Gamma_{54}}{\alpha} \delta(s)
\end{align*}
\]

where

\[\alpha = \Gamma_{45} \Gamma_{54} - \Gamma_{44} \Gamma_{55}\]

so that application of the inverse Radon-transform will lead to

\[
\begin{align*}
U_{44} &= -\frac{1}{8\pi^2} \int_{|n|=1} \partial_x^2 \hat{U}_{44} \, dS(n) = \frac{1}{16\pi^2} \int_{|n|=1} \alpha_{44} \partial_x \frac{E^q_{ij}}{\rho c^2_q E^q_{pp}} \left\{ 2\delta(n \cdot x) + ik_q \right\} e^{ik_q |n| x} \, dS(n) - \frac{1}{8\pi^2} \int_{|n|=1} \frac{\Gamma_{55}}{\alpha} \delta(n \cdot x) \, dS(n) \\
U_{54} &= -\frac{1}{8\pi^2} \int_{|n|=1} \partial_y^2 \hat{U}_{54} \, dS(n) = \frac{1}{16\pi^2} \int_{|n|=1} \alpha_{54} \partial_y \frac{E^q_{ij}}{\rho c^2_q E^q_{pp}} \left\{ 2\delta(n \cdot x) + ik_q \right\} e^{ik_q |n| x} \, dS(n) + \frac{1}{8\pi^2} \int_{|n|=1} \frac{\Gamma_{54}}{\alpha} \delta(n \cdot x) \, dS(n)
\end{align*}
\]

Eqs. (49)–(51) and (55)–(57) are the 3-D dynamic time-harmonic magnetoelectroelastic Green’s functions, which have the following symmetry property

\[U_{KM}(x, \omega) = U_{MK}(x, \omega)\]

The expressions for the displacement Green’s functions may be recast into a compact form as

\[\varepsilon_{KM}^q = \begin{cases} 
E_{km}^q & K, M = 1, 2, 3 \\
\alpha_{KM}^q E_{lm}^q & K = 4, 5; \quad M = 1, 2, 3 \\
\alpha_{LM}^q E_{ij}^q & K, M = 4, 5
\end{cases}\]

and
\[ A_{KM} = \frac{1}{\xi} \left\{ \Gamma_{KM} (\delta_{4K} \delta_{SM} + \delta_{5K} \delta_{AM}) - \frac{\Gamma_{44} \Gamma_{55}}{T_{KM}} (\delta_{4K} \delta_{4M} + \delta_{5K} \delta_{5M}) \right\} \]

The obtained time-harmonic Green’s functions may be split into a singular static part plus a regular frequency dependent part as

\[ U_{KM}(\mathbf{x}, \omega) = U^{S}_{KM}(\mathbf{x}) + U^{R}_{KM}(\mathbf{x}, \omega) \tag{61} \]

where

\[ U^{S}_{KM}(\mathbf{x}) = \frac{1}{8\pi^2} \int_{|\mathbf{n}|=1} \frac{e_{\xi \eta}^{\xi \eta} \delta(\mathbf{n} \cdot \mathbf{x}) dS(\mathbf{n})}{\rho \varepsilon^{\xi \eta} \varepsilon^{\eta \xi}} + \frac{1}{8\pi^2} \int_{|\mathbf{n}|=1} A_{KM} \delta(\mathbf{n} \cdot \mathbf{x}) dS(\mathbf{n}) \tag{62} \]

\[ U^{R}_{KM}(\mathbf{x}, \omega) = \frac{i}{16\pi^2} \int_{|\mathbf{n}|=1} \frac{k_{\xi \eta}^{\xi \eta} e^{i\omega|\mathbf{n}|}}{\rho \varepsilon^{\xi \eta} \varepsilon^{\eta \xi}} dS(\mathbf{n}) \tag{63} \]

The static Green’s functions are singular and can be reduced to explicit expressions, which are given in Appendix B following the work by Pan (2002). The above decomposition is useful for BEM implementation purposes (see, e.g., Denda et al., 2004 or Sáez et al., 2006).

### 3.2. 2-D time-harmonic Green’s functions

The 2-D time-harmonic Green’s functions may be obtained following the same procedure as for the 3-D case. In 2-D case the lowercase (elastic) subscripts take values 1 and 2 only, whilst the uppercase (extended) subscripts take values 1, 2 (elastic), 4 (electric) and 5 (magnetic). In this way, Green’s functions are obtained in the form of line integrals over a unit circle \(|n| = 1\) as (see Wang and Achenbach, 1994 or Wang and Zhang, 2005 for details)

\[ U_{KM}(\mathbf{x}, \omega) = U^{S}_{KM}(\mathbf{x}) + U^{R}_{KM}(\mathbf{x}, \omega) \tag{64} \]

where

\[ U^{S}_{KM}(\mathbf{x}) = -\frac{1}{4\pi^2} \int_{|\mathbf{n}|=1} \frac{e_{\xi \eta}^{\xi \eta} \log |\mathbf{n} \cdot \mathbf{x}| dL(\mathbf{n})}{\rho \varepsilon^{\xi \eta} \varepsilon^{\eta \xi}} - \frac{1}{4\pi^2} \int_{|\mathbf{n}|=1} A_{KM} \log |\mathbf{n} \cdot \mathbf{x}| dL(\mathbf{n}) \tag{65} \]

\[ U^{R}_{KM}(\mathbf{x}, \omega) = \frac{1}{16\pi^2} \int_{|\mathbf{n}|=1} \frac{e_{\xi \eta}^{\xi \eta} \Phi^{\xi \eta}(k_q, |\mathbf{n} \cdot \mathbf{x}|)}{\rho \varepsilon^{\xi \eta} \varepsilon^{\eta \xi}} dL(\mathbf{n}) \tag{66} \]

in which \(e_{\xi \eta}^{\xi \eta}\) is given by (60) and

\[ \Phi^{\xi \eta}(k_q, |\mathbf{n} \cdot \mathbf{x}|) = \phi(k_q |\mathbf{n} \cdot \mathbf{x}|) + 2 \log(|\mathbf{n} \cdot \mathbf{x}|) \tag{67} \]

with

\[ \phi(\zeta) = i\pi e^{i\eta} - 2[\cos(\zeta)ci(\zeta) + \sin(\zeta)si(\zeta)] \tag{68} \]

and \(ci\) and \(si\) are, respectively, the cosine integral and the sine integral defined by

\[ ci(\zeta) = -\int_{\zeta}^{\infty} \frac{\cos z}{z} \, dz; \quad si(\zeta) = -\int_{\zeta}^{\infty} \frac{\sin z}{z} \, dz \tag{69} \]

As in 3-D, explicit expressions for the 2-D static magnetoelastic Green’s functions may be obtained (see Jiang and Pan, 2004 for details), which are given in Appendix C.

Once the 2-D and the 3-D displacement Green’s functions \(U_{KM}\) have been obtained, the corresponding traction Green’s functions \(P_{KM}\) needed for a BEM implementation can be immediately derived by substitution of \(U_{KM}\) into the generalized Hooke’s law to yield

\[ P_{MK} = \eta_i C_{ijkl} U_{jk,l} \tag{70} \]

where \(\eta_i\) are the components of the external unit normal vector to the boundary at the observation point where Green’s functions are being evaluated.
The displacement Green’s functions $U_{KM}$ have the following weak singularities

$$U_{KM} = \begin{cases} \frac{1}{r}, & r \to 0 \ (3-D) \\ \log(r), & r \to 0 \ (2-D) \end{cases}$$

while the traction Green’s functions have the following Cauchy-type strong singularities

$$P_{KM} = \begin{cases} \frac{1}{r^2}, & r \to 0 \ (3-D) \\ \frac{1}{r}, & r \to 0 \ (2-D) \end{cases}$$

in which $r = |x|$. In the numerical implementation of BEM, special attention must be paid to the singular behaviors of the time-harmonic Green’s functions.

It can be easily verified that the obtained 2-D and 3-D magnetoelectroelastic Green’s functions reduce to the piezoelectric fundamental solutions given by Wang and Zhang (2005) when the piezomagnetic and electromagnetic constants vanish.

<table>
<thead>
<tr>
<th>$C_{11}$ (GPa)</th>
<th>$C_{12}$ (GPa)</th>
<th>$C_{22}$ (GPa)</th>
<th>$C_{66}$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>226</td>
<td>124</td>
<td>216</td>
<td>44</td>
</tr>
<tr>
<td>$\varepsilon_{21}$ (C/m²)</td>
<td>$\varepsilon_{22}$ (C/m²)</td>
<td>$\varepsilon_{16}$ (C/m²)</td>
<td>$\varepsilon_{15}$ (C/m²)</td>
</tr>
<tr>
<td>-2.2</td>
<td>9.3</td>
<td>-12</td>
<td>5.8</td>
</tr>
<tr>
<td>$h_{21}$ (N/A m)</td>
<td>$h_{22}$ (N/A m)</td>
<td>$h_{16}$ (N/A m)</td>
<td>$h_{15}$ (N/A m)</td>
</tr>
<tr>
<td>350</td>
<td>2737.5</td>
<td>83.5</td>
<td>297</td>
</tr>
</tbody>
</table>

Table 1: Material properties of BaTiO₃–CoFe₂O₄ with volume fraction $V_f = 0.5$

Fig. 1. 2-D $U_{22}$ extended displacement in BaTiO₃–CoFe₂O₄ versus dimensionless frequency for points along the $x_2$-axis.
4. Numerical evaluation of Green’s functions and BEM formulation

The obtained Green’s functions or fundamental solutions are relevant for BEM-based applications (Domínguez, 1993). The displacement boundary integral representation for a collocation point $\xi$ in an linear magneto electroelastic domain $\mathcal{D}$ with boundary $\Gamma$ subjected to time-harmonic loading follows from application of the generalized reciprocal theorem between the state defined by the fundamental solution and the state under study, leading to

$$\int_{\Gamma} \mathbf{u}(\xi) \cdot \mathbf{g}(\xi, \eta) \, d\Gamma = \mathbf{u}_0(\eta),$$

where $\mathbf{u}(\xi)$ is the displacement at the collocation point, $\mathbf{g}(\xi, \eta)$ is the fundamental solution, and $\mathbf{u}_0(\eta)$ is the state under study.

![Graph](https://via.placeholder.com/150)

**Fig. 2.** 2-D $U_{24}$ extended displacement in BaTiO$_3$–CoFe$_2$O$_4$ versus dimensionless frequency for points along the $x_2$-axis.

![Graph](https://via.placeholder.com/150)

**Fig. 3.** 2-D $U_{25}$ extended displacement in BaTiO$_3$–CoFe$_2$O$_4$ versus dimensionless frequency for points along the $x_2$-axis.
\[ c_{IJ}(\xi)u_J(\xi, \omega) + \int_{\Gamma} P_{IJ}(\mathbf{x}, \xi, \omega)u_J(\mathbf{x}, \omega)\,d\Gamma(\mathbf{x}) = \int_{\Gamma} U_{IJ}(\mathbf{x}, \xi, \omega)p_J(\mathbf{x}, \omega)\,d\Gamma(\mathbf{x}) \]

where \( \omega \) is the angular frequency of excitation; \( U_{IJ} \) and \( P_{IJ} \) are the fundamental solution displacements and tractions, respectively; and \( c_{IJ}(\xi) \) are the so-called free terms that result from the Cauchy principal value (CPV) integration of the strongly singular \( P_{IJ} \) kernels.

Fig. 4. 2-D \( U_{44} \) extended displacement in BaTiO\(_3\)-CoFe\(_2\)O\(_4\) versus dimensionless frequency for points along the \( x_2 \)-axis.

Fig. 5. 2-D \( U_{55} \) extended displacement in BaTiO\(_3\)-CoFe\(_2\)O\(_4\) versus dimensionless frequency for points along the \( x_2 \)-axis.
One of the major concerns when dealing with Eq. (73) is to adequately evaluate the obtained Green’s functions, since they are not given in explicit form but as line/surface integrals over a unit circumference/sphere. Provided that the mathematical structure of the fundamental solution given in this paper is similar to that of the solutions previously developed by other authors for both anisotropic (Wang and Achenbach, 1994; Wang and Achenbach, 1995) and piezoelectric (Denda et al., 2004; Wang and Zhang, 2005) solids using the Radon-transform, the techniques already implemented to cast such Green’s functions into BEM codes in an efficient way may be extended in a straightforward manner for magnetoelectroelasticity (see Denda et al., 2004; Sáez and Domínguez, 1999; Sáez et al., 2006; Wang et al., 1996 for details).

However, when far-field or high-frequency applications come into play, numerical evaluation of the obtained Green’s functions becomes costly – from a computational point of view – due to the oscillatory nature of the integrands. Therefore development of asymptotic expressions to cover such cases becomes mandatory in order to have a reasonable BEM code. This can be achieved for instance by extending the work presented by Sáez and Domínguez (2000) for anisotropic materials to the magnetoelectroelastic case.

Finally, computations for the 2-D Green’s functions have been carried out to illustrate the magnetoelectroelastic coupled behavior of a BaTiO$_3$–CoFe$_2$O$_4$ composite material with a volume fraction $V_f = 0.5$. Material constants are listed in Table 1 (Song and Sih, 2003). The magnetic and electric poling directions of the material coincide with the $x_2$ axis. The $U_{22}$ component of the displacement Green’s functions are plotted in Fig. 1 versus the dimensionless frequency $\omega r/c_s$, $r$ being the distance between the origin where the point load is applied and points located on the $x_2$ axis where the displacement $U_{22}$ is computed and $c_s = \sqrt{C_{66}/\rho}$. Similarly, the $U_{24}$, $U_{25}$, $U_{44}$, $U_{55}$ and $U_{45}$ components of the displacement Green’s functions are plotted in Figs. 2–6.

5. Conclusions

In this paper dynamic time-harmonic Green’s functions for 2-D and 3-D linear fully anisotropic magnetoelectroelastic solids have been derived via Radon-transform. The resulting Green’s functions have been expressed as surface integrals over a unit sphere in 3-D case and line integrals over a unit circle in 2-D case. Furthermore, the obtained fundamental solutions may be split into a singular static part plus a regular dynamic part. The singular static part corresponds to the static magnetoelectroelastic Green’s functions and the regular dynamic part consists of continuous functions. The singular static Green’s functions can be...
reduced to closed form expressions. The simple and bounded structure of the integral expressions for the
dynamic part of the Green’s functions make them suitable for a numerical implementation into a BEM code,
as previously illustrated for anisotropic elastic and piezoelectric solids (Denda et al., 2004; Sáez and Domínguez,
1999; Sáez et al., 2006; Wang et al., 1996). As it could be expected, the obtained Green’s functions reduce
to the existing solutions for piezoelectric solids (Wang and Zhang, 2005) when the piezomagnetic and electromag-
netic constants vanish.

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Appendix A. Radon-transform

The Radon-transform of an arbitrary function \( f(x) \) is defined by

\[
\hat{f}(s, \mathbf{n}) = R\{ f(x) \} = \int_{|\mathbf{n}| = 1} f(x) \delta(s - \mathbf{n} \cdot \mathbf{x}) \, d\mathbf{x},
\]
where \( s = \mathbf{n} \cdot \mathbf{x} \) is a real transform parameter and \( \mathbf{n} \) is a unit normal vector. The Radon-transform is an inte-
gration of \( f(x) \) over \( \mathbf{n} \cdot \mathbf{x} = s \), i.e., over a surface for 3-D and along a line for 2-D.

The inverse Radon-transform is given by

\[
f(x) = R^*\{ \hat{f}(s, \mathbf{n}) \} = \int_{|\mathbf{n}| = 1} \hat{f}(\mathbf{n} \cdot \mathbf{x}, \mathbf{n}) \, d\mathbf{n},
\]

where

\[
\hat{f}(s, \mathbf{n}) = \mathcal{K}\{ \hat{f}(s, \mathbf{n}) \} = \begin{cases} 
-\frac{1}{8\pi^2} \partial_s^2 \hat{f}(s, \mathbf{n}), & \text{for 3-D} \\
\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\delta(s - \mathbf{n} \cdot \mathbf{x})}{s - \sigma} \, d\sigma, & \text{for 2-D}
\end{cases}
\]

The inverse Radon-transform \( R^* \) defined by Eq. (A.2) is a surface integral over a unit sphere in 3-D case and a
line integral over a unit circle in 2-D case.

Appendix B. 3-D static magnetoelectroelastic Green’s functions

Following essentially the same procedure as for the anisotropic (Wang and Achenbach, 1994) and the pie-
zoelectric (Wang and Zhang, 2005) solids an explicit solution can be obtained for magnetoelectroelastic static
Green’s functions (see Pan, 2002 for details).

The Eq. (21), disregarding inertial terms, admits an integral solution in the form

\[
U^S_{KM}(\mathbf{x}) = \frac{1}{8\pi^2} \int_{\Omega} \Gamma^{-1}_{KM}(\mathbf{n}) \delta(\mathbf{n} \cdot \mathbf{x}) \, d\Omega(\mathbf{n})
\]

where \( \Omega \) represents any closed surface around the origin \( \mathbf{n} = \mathbf{0} \) and \( \Gamma_{KM} \) is the generalized Christoffel tensor
defined in (23). Note here that Eq. (B.1) is identical to the integral expression (62).

Taking a suitable orthonormal base, say \( (\mathbf{e}_j = \mathbf{x}/|\mathbf{x}|, \mathbf{p}, \mathbf{q}) \), and using the residue theorem, an explicit expres-
sion can be reached for this integral as

\[
U^S_{KM}(\mathbf{x}) = -\frac{1}{2\pi r} \Im\left( \sum_{J=1}^{J=5} \frac{\text{Adj}[\Gamma_{KM}(\mathbf{p} + \mu_J \mathbf{q})]}{a_{10}(\mu_J - \bar{\mu}_J) \prod_{l=1, l \neq J} (\mu_J - \mu_l)(\mu_J - \bar{\mu}_l)} \right)
\]

where \( r = |\mathbf{x}|, \mathcal{I}(\cdot) \) stands for the imaginary part, overline indicates complex conjugate, \( \mu_J \) are the roots of the
characteristic equation of the material with positive imaginary part, i.e.,
\[ \det[\Gamma_{KM}(p + \mu_j q)] = 0, \quad J = 1, \ldots, 5, \]
and \(a_{10}\) is the coefficient of the 10th term of the 10th order polynomial \(\det[\Gamma_{KM}(p + \mu_j q)]\).

**Appendix C. 2-D static magnetoelectroelastic Green's functions**

Similarly, the static displacement Green's functions for 2-D problems can be written as

\[ U_{KM}^{S}(x) = -\frac{1}{4\pi^2} \int_{\Omega} \frac{\Gamma_{KM}^{-1}(n)}{n \cdot x} \log |n \cdot x| \omega(n) \]

where \(\Omega\) is any closed 2-D curve around the origin \(n = 0\) and

\[ \omega(n) = n_1 d_n - n_2 d_n \]

It can be shown that Eq. (C.1) is identical to the integral expression (65).

Following the procedure for piezoelectric solids (Denda et al., 2004; Wang and Zhang, 2005), an explicit expression for the displacement Green’s functions can be obtained as (see Jiang and Pan, 2004 for details)

\[ U_{KM}^{S}(x) = \frac{1}{\pi} \mathcal{I} \left( \sum_{j=1}^{5} \frac{\operatorname{Adj}[\Gamma_{KM}(1, \mu_j)]}{\partial_{\mu_j} \det[\Gamma_{KM}(1, \mu_j)]} \log \frac{|z_j + i\mu_j|}{\mu_j + 1} \right) \]

where

\[ z_j = x_1 + \mu_j x_2, \quad \mathcal{I}(\mu_j) > 0 \]

and \(\mu_j\) are the roots of the material characteristic equation \(\det[\Gamma_{KM}(1, \mu)] = 0\).

**References**


