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Integral Equations without a Unique Solution can be made Useful for Solving some Plane Harmonic Problems †

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Using Green's third identity an integral equation for a two dimensional harmonic problem is derived. For a particular exceptional geometry the integral equation does not have a unique solution but by applying Green's third identity a supplementary integral condition is derived. When the integral equation and the integral condition are solved simultaneously we obtain always a unique solution. The procedure is demonstrated by some numerical examples.

1. Introduction

IN CONNECTION with some problems within electrochemical machining (ECM) (Rasmussen & Christiansen, 1973) one encounters a two dimensional potential problem in a doubly-connected region. The boundary conditions are very simple, but this is not the case with the form of the boundary curves. In such a case it seems best to use an integral equation method, namely the integral equation which is derived here (Section 2). In some exceptional cases this integral equation does not have a unique solution; this surprising fact can be shown by quite elementary means (Section 3). The integral equation is derived using Green's third identity which is considered (Section 4) in order to find whether it also has surprising properties. An elementary investigation (Section 5) shows that this is so. When these surprising facts are traced back to Green's third identity, we solve (Section 6) the problem by considering the identity more closely and derive a supplementary condition which the solution of the integral equation has to satisfy in the case when it is non-unique. The supplementary condition can be combined with the integral equation (Section 7) also when it is unnecessary. By means of numerical examples it is demonstrated (Section 8) that if the integral equation is solved without the supplementary condition then the results are grossly wrong, while the addition of the supplementary condition, which is easy to do, effectively eliminates the exceptional geometry. Finally the results are discussed (Section 9).

2. Derivation of the Integral Equation

Let Γ_0 and Γ_1 be two plane, simple, and closed curves without double points. The exterior curve Γ_0 surrounds the interior curve Γ_1. They do not touch each other. The

† A very short version of this paper has been presented on 4 April 1973 the Scientific Meeting of "Gesellschaft für Angewandte Mathematik und Mechanik" (GAMM), which was held in Munich (Germany): cf. Christiansen (1974).

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ringshaped domain between the two curves is denoted by D. Here the following two dimensional potential problem is given: We seek a function u which satisfies

$$\Delta u = 0 \qquad \text{in } D \tag{2.1}$$

with the boundary conditions

$$u = u_0 \qquad \text{(a constant) on } \Gamma_0, \tag{2.2a}$$

$$u = u_1 \qquad \text{(a constant) on } \Gamma_1. \tag{2.2b}$$

See Fig. 1.

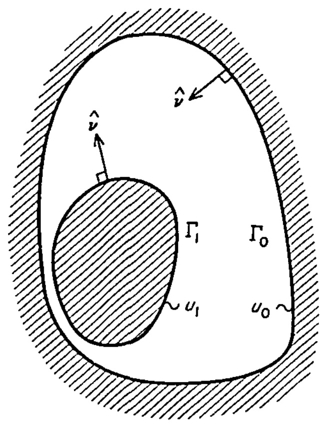

FIG. 1. An external curve Γ_0 with the potential u_0 and an internal curve Γ_1 with the potential u_1. The direction of the normals to both curves are *inwards* to the ring-shaped domain.

For solving this problem an integral equation of the second kind can be used (Mikhlin, 1964, Section 31), but due to the multi-connectivity special problems come up. Furthermore the unknown of the integral equation does not have a simple connection with the normal derivative of u on the boundaries $\Gamma_0 \cup \Gamma_1$, which in the physical problem considered is of particular interest.

However we shall here—by quite elementary means—derive an integral equation of the first kind where the unknown is equal to the normal derivative just mentioned.

The basis for our derivation of the integral equation is Green's third identity (Courant & Hilbert, 1962, pp. 256–257). It expresses the value of an harmonic function u at a point $\bar{\mathbf{r}}'$ in a domain by an integral along a curve Γ enclosing this domain, where Γ is simple and consists of a finite number of smooth curves:

$$\frac{1}{2\pi}\int_{\Gamma}\left\{\ln|\bar{\mathbf{r}}-\bar{\mathbf{r}}'|\,\frac{\partial u}{\partial \nu}(\bar{\mathbf{r}})-\frac{\partial}{\partial \nu}\ln|\bar{\mathbf{r}}-\bar{\mathbf{r}}'|\,u(\bar{\mathbf{r}})\right\}ds = u(\bar{\mathbf{r}}'). \tag{2.3}$$

Here $\partial/\partial\nu$ indicates differentiation with respect to $\bar{\mathbf{r}}$ in the normal direction *inward* from Γ.

As Γ we here choose $\Gamma_0 \cup \Gamma' \cup \Gamma_1 \cup \Gamma''$, where Γ' and Γ'' are two closely spaced parallel curves connecting Γ_0 and Γ_1. Since u is harmonic in the ringshaped domain D, the contribution of the integral along $\Gamma' \cup \Gamma''$ is zero, and (2.3) can be used with $\Gamma = \Gamma_0 \cup \Gamma_1$, and $\bar{\mathbf{r}}' \in D$. Application of the boundary conditions (2.2), and the fact

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that

$$-\frac{1}{2\pi}\int_{\left\{\Gamma_0 \atop \Gamma_1\right\}} \frac{\partial}{\partial \nu} \ln|\bar{\mathbf{r}}-\bar{\mathbf{r}}'|\, ds = \begin{Bmatrix}1\\0\end{Bmatrix}, \qquad \bar{\mathbf{r}}' \in D$$

gives us

$$u_0+\frac{1}{2\pi}\int_{\Gamma_0\cup\Gamma_1} \ln|\bar{\mathbf{r}}-\bar{\mathbf{r}}'|\, \frac{\partial u}{\partial \nu}(\bar{\mathbf{r}})\, ds \equiv u(\bar{\mathbf{r}}'), \qquad \bar{\mathbf{r}}' \in D.$$

In this identity the point $\bar{\mathbf{r}}'$ is now moved to either of the curves. The limit process is also carried out inside the integral, but this does not give rise to difficulties since there is only a logarithmic kernel. Now using (2.2)

$$\bar{\mathbf{r}}' \to \bar{\mathbf{r}}_0 \in \Gamma_0 \Rightarrow u(\bar{\mathbf{r}}') \to u(\bar{\mathbf{r}}_0) = u_0$$
$$\bar{\mathbf{r}}' \to \bar{\mathbf{r}}_0 \in \Gamma_1 \Rightarrow u(\bar{\mathbf{r}}') \to u(\bar{\mathbf{r}}_0) = u_1$$

the following integral equation is obtained

$$\int_{\Gamma_0\cup\Gamma_1} \ln|\bar{\mathbf{r}}-\bar{\mathbf{r}}_0|\,.\,\psi(\bar{\mathbf{r}})\, ds = \begin{Bmatrix}0, & \bar{\mathbf{r}}_0 \in \Gamma_0\\ 2\pi(u_1-u_0), & \bar{\mathbf{r}}_0 \in \Gamma_1\end{Bmatrix}, \tag{2.4}$$

where the function ψ is

$$\psi(\bar{\mathbf{r}}) = \frac{\partial u}{\partial \nu}(\bar{\mathbf{r}}) \tag{2.5}$$

where $\partial/\partial\nu$ denotes differentiation with respect to the integration point $\bar{\mathbf{r}}$ in the normal direction of Fig. 1.

For general Dirichlet boundary conditions a similar integral equation is known (Jaswon, 1963, pp. 28–29), but in this case it is not possible to identify the unknown as we have done here (2.5). This identification is essential for our practical application of the integral equation. Therefore we have carried out the derivation.

3. Non-uniqueness of the Solution

The Dirichlet problem defined in (2.1) and (2.2) has a unique solution (see e.g. Muschelischwili, 1965, IIIA). But the corresponding integral equation (2.4) does not always have a unique solution: there exists namely *always* an exceptional case where (2.4) does *not* have a unique solution. This exceptional case depends solely on the *exterior* curve Γ_0, in that for a given *shape* of Γ_0 there exists a particular *magnitude* of Γ_0, where (2.4) does not have a unique solution (Jaswon, 1963, pp. 27–29).

This—somewhat surprising—result can be shown convincingly in the case when the curves Γ_0 and Γ_1 are two concentric circles C_0 and C_1 with radii c_0 and c_1, respectively. Due to the symmetry the unknown will attain constant values on each of the two curves; we denote the values by ψ_0 and ψ_1, respectively. The equation (2.4) is then transformed to

$$\psi_0 \int_{C_0} \ln|\bar{\mathbf{r}}-\bar{\mathbf{r}}_0|\, ds+\psi_1 \int_{C_1} \ln|\bar{\mathbf{r}}-\bar{\mathbf{r}}_0|\, ds = 0; \qquad \bar{\mathbf{r}}_0 \in C_0,$$

$$\psi_0 \int_{C_0} \ln|\bar{\mathbf{r}}-\bar{\mathbf{r}}_0|\, ds+\psi_1 \int_{C_1} \ln|\bar{\mathbf{r}}-\bar{\mathbf{r}}_0|\, ds = 2\pi(u_1-u_0); \quad \bar{\mathbf{r}}_0 \in C_1.$$

Introducing the arc-parameter descriptions

$$C_0\colon ds = c_0\, d\theta_0, \qquad C_1\colon ds = c_1\, d\theta_1$$

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gives

$$\psi_0 c_0 \int_0^{2\pi} \ln \rho \, d\theta_0 + \psi_1 c_1 \int_0^{2\pi} \ln \rho \, d\theta_1 = 0; \qquad \bar{\mathbf{r}}_0 \in C_0,$$

$$\psi_0 c_0 \int_0^{2\pi} \ln \rho \, d\theta_0 + \psi_1 c_1 \int_0^{2\pi} \ln \rho \, d\theta_1 = 2\pi(u_1 - u_0); \quad \bar{\mathbf{r}}_0 \in C_1,$$

where $\rho = |\bar{\mathbf{r}} - \bar{\mathbf{r}}_0|$. Here the integrals can be evaluated in closed form using (A.3) giving the following system of two linear algebraic equations with the unknowns ψ_0 and ψ_1:

$$\psi_0 c_0 \ln c_0 + \psi_1 c_1 \ln c_0 = 0 \tag{3.1a}$$

$$\psi_0 c_0 \ln c_0 + \psi_1 c_1 \ln c_1 = u_1 - u_0. \tag{3.1b}$$

The determinant of this system is

$$\Delta = c_0 c_1 \ln(c_1/c_0) \ln c_0.$$

It is seen that $\Delta = 0$ if and only if $c_0 = 1$ independent of c_1; $c_0 > c_1 > 0$. Thus it is *solely* (the radius of) the exterior curve which determines whether the exceptional geometry occurs, in which case the solution ψ_0 cannot be determined uniquely.

About non-uniqueness for integral equations for *two dimensional* potential problems the following can be said.

(a) Muschelischwili treats the Dirichlet problem (1965, IIIA) and shows that by solving such problems by means of integral equations with logarithmic kernel there always appears an *exceptional* case ("Ausnahmefall") where the integral equations break down (ibid., Section 65, pp. 239–241).

(b) Jaswon (1963, p. 27) has shown that for a given curve-shape there will always exist a particular curve-magnitude, where difficulties will occur in connection with the integral equations of the first kind with logarithmic kernel.

(c) Harrington *et al.* (1969) have empirically found that numerically difficulties appear if the boundary curve is a circle with radius equal to one, but no difficulties were observed in connection with curves other than circles (ibid., Section 4, p. 1717, right column). The reason is possibly that with a circle the quadrature error in transforming the integral equation into a set of linear algebraic equations is small, while this error is greater when other curves are considered. The greater quadrature error disturbs the picture and may well camouflage the singularity.

(d) Hayes & Kellner (1972) have very thoroughly investigated the integral equations for solving plane potential-problems, and they have shown that the integral equation can have an eigenvalue equal to zero, if the transfinite diameter of the considered boundary curve is equal to one. For a doubly connected region it is the transfinite diameter of the exterior boundary curve which is crucial (ibid., Section 5).

On "transfinite diameter" or "exterior (mapping) radius", see e.g. Pólya & Szegö (1931, Section 1; 1951) and Hille (1962, Sections 16 and 17). We denote the transfinite diameter by d. For a circle with radius c (Pólya & Szegö, 1931, p. 8):

$$\text{circle: } d = c, \tag{3.2a}$$

which shows that the exceptional case appears when $c = 1$, in accordance with the circle-example given above. For use in Section 8 we notice that for an ellipse with semiaxes a and b (ibid., p. 9):

$$\text{ellipse: } d = \tfrac{1}{2}(a+b). \tag{3.2b}$$

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The exceptional geometry gives rise to practical difficulties. In the references cited in (a), (b) and (d) above it is recommended to avoid the difficulties simply by changing the scale of the given geometry so that the curve is away from the critical magnitude. But nothing is said there about the value of the critical magnitude for a given shape of the curve, and if the critical magnitude were known nothing is said about whether it is necessary to keep a certain "security-distance" from the critical magnitude.† By this we mean that from a theoretical point of view there is a particular critical magnitude, but from a practical-computational point of view the question arises as to whether there is a certain zone where the magnitudes are more or less critical and, if so, what is the size of this zone? Further there are some cases (Rasmussen & Christiansen, 1973) where one has to consider a series of curves with different magnitudes which makes a choice of scaling troublesome.

4. Green's Third Identity

The integral equation (2.4), derived by means of Green's third identity, has some strange properties, i.e. an exceptional geometry exists. Therefore we shall here consider Green's third identity more closely to find whether the identity has similar characteristics.

Let Γ be a plane, closed, and simple curve, which is smooth, i.e. which has a continuous tangent at each point. Then for an harmonic function, $u(\bar{\mathbf{r}})$, depending upon *two* space variables, $\bar{\mathbf{r}} = (x, y)$, Green's third identity (Courant & Hilbert, 1962, pp. 256–257) states that

$$\int_\Gamma \left\{\frac{\partial u}{\partial \nu}(\bar{\mathbf{r}})G(\bar{\mathbf{r}}', \bar{\mathbf{r}}) - u(\bar{\mathbf{r}})\frac{\partial G}{\partial \nu}(\bar{\mathbf{r}}', \bar{\mathbf{r}})\right\} ds = \begin{cases} 0\cdot u(\bar{\mathbf{r}}'), \bar{\mathbf{r}}' & \text{outside } \Gamma \\ \frac{1}{2}\cdot u(\bar{\mathbf{r}}'), \bar{\mathbf{r}}' & \text{on } \Gamma \\ 1\cdot u(\bar{\mathbf{r}}'), \bar{\mathbf{r}}' & \text{inside } \Gamma \end{cases}, \qquad \begin{matrix} (4.1(0)) \\ (4.1(\tfrac{1}{2})) \\ (4.1(1)) \end{matrix}$$

where the elementary solution of the equation $\Delta u = 0$ in two dimensions is

$$G(\bar{\mathbf{r}}', \bar{\mathbf{r}}) = \frac{1}{2\pi}\ln\frac{1}{\rho}, \tag{4.2a}$$

$$\rho = |\bar{\mathbf{r}} - \bar{\mathbf{r}}'|. \tag{4.2b}$$

Here by $\partial/\partial\nu$ we mean differentiation with respect to $\bar{\mathbf{r}}$ in the direction of the *outward* normal to the curve Γ at the point of integration $\bar{\mathbf{r}}$ on Γ.

Under the sign of integration in (4.1) appear the values of u and $\partial u/\partial\nu$ on Γ; if the correct corresponding boundary values of u and $\partial u/\partial\nu$ are known everywhere on Γ, then the value of u everywhere inside Γ can be computed by means of (4.1(1)). When Green's third identity is used as a basis for derivation of integral equations (or functional equations), one should at this stage pose the following "opposite" question: can (4.1) be considered as a compatibility equation which (in all three regions) connects the boundary values of u and $\partial u/\partial\nu$? The answer is divided into three parts.

Case (0): $\bar{\mathbf{r}}'$ *outside* Γ, *i.e.* (4.1(0))

This formula gives a (not quite obvious) connection between the boundary values which has been used by Kupradze for deriving functional equations (1963, Ch. VII,

† Based upon empirical-numerical investigations Harrington *et al.* (1969, Section 2, p. 1715) recommend applying a scale-factor, expressed by the maximum distance between two points on the boundary curve.

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Sections 16–17; 1965, Ch. X, Sections 15–17) or the method of generalized Fourier series (1965, Ch. X, Sections 20–38; 1969, Section 2).

Case ($\frac{1}{2}$): $\bar{\mathbf{r}}'$ *on* Γ, *i.e.* (4.1($\frac{1}{2}$))

This formula—Green's Boundary Formula—gives the clearest connection between the boundary values, and it has been used for the derivation of integral equations to solve the mixed boundary value problem (Jaswon, 1963, pp. 29–30; Bhargava & Saxena, 1971, p. 248). Specifically a Dirichlet-problem (u prescribed) can be solved in that $\partial u/\partial v$ becomes the unknown in an integral equation of the first kind with logarithmic kernel; cf. the problem of Section 2. Further a Neumann-problem ($\partial u/\partial v$ prescribed) can be solved in that u becomes the unknown in an integral equation of the second kind. Such an equation appears by solution of Saint–Venant's torsion problem (Sokolnikoff, 1956, Sections 34–35) by means of the harmonic "warping function" or "torsion function". This integral equation can be solved numerically (Jaswon & Ponter, 1963; Kandler, 1967).

Case (1): $\bar{\mathbf{r}}'$ *inside* Γ, *i.e.* (4.1(1))

This formula can be used in the following way to establish a connection between the boundary values. Introduce two "arbitrary" functions as "boundary values" for u and $\partial u/\partial v$ in (4.1), evaluate the integral—which we denote by $f(\bar{\mathbf{r}}')$—expressed by the two introduced functions, determine the limiting value:

$$\lim_{\bar{\mathbf{r}}' \to \bar{\mathbf{r}}_0 \in \Gamma} f(\bar{\mathbf{r}}'),$$

and formulate the requirement that this limiting value is equal to the value (in the point $\bar{\mathbf{r}}_0$) of the "arbitrary" function u, which was introduced into (4.1). This gives rise to a "connection" between the two "arbitrary" "boundary-value" functions u and $\partial u/\partial v$. This last way of deriving compatibility equations is the most complicated, but it seems to be the most powerful when the ideas have to be generalized to other and more complicated problems.

5. Green's Third Identity for a Circle

The investigation of the question whether (4.1) gives a connection between the boundary values u and $\partial u/\partial v$, cf. the three cases in Section 4, is here carried out in an elementary manner. To this end the arbitrary boundary curve Γ may be chosen as the particular boundary curve: a circle C with radius c:

$$C\colon x = c\cos\theta,\ y = c\sin\theta; \qquad 0 \leqslant \theta \leqslant 2\pi.$$

Using polar coordinates the integration point $\bar{\mathbf{r}}$ is characterized by (c, θ), while the parameter point $\bar{\mathbf{r}}'$ is characterized by (R, ϕ), where $0 \leqslant \phi \leqslant 2\pi$, $0 \leqslant R < \infty$. The definite integral in (4.1) becomes then a function $f = f(R, \phi)$, which by simple geometrical considerations can be written ($\partial/\partial v = \partial/\partial r$):

$$f(R, \phi) = \frac{c}{2\pi}\int_0^{2\pi}\left\{(c - R\cos(\theta-\phi))\rho^{-2}\ u(c, \theta) - \ln\rho\ \frac{\partial u}{\partial r}(c, \theta)\right\} d\theta \qquad (5.1a)$$

where, cf. (4.2b),

$$\rho^2 = c^2 + R^2 - 2cR\cos(\theta-\phi). \qquad (5.1b)$$

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The boundary values for u and $\partial u/\partial r$ are expressed as Fourier-series

$$u(c, \theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta), \tag{5.2a}$$

$$\frac{\partial u}{\partial r}(c, \theta) = c^{-1}\left[A_0 + \sum_{n=1}^{\infty} n(A_n \cos n\theta + B_n \sin n\theta)\right]. \tag{5.2b}$$

The series (5.2) are inserted into the integral (5.1) and after elementary calculations, and application of the integrals (A1), (A2), (A3), and (A4), one obtains that $f(R, \phi)$, (5.1) is

$$f(R, \phi) = \begin{Bmatrix} 0 - A_0 \ln R \\ \tfrac{1}{2}a_0 - A_0 \ln c \\ a_0 - A_0 \ln c \end{Bmatrix} +$$

$$\tfrac{1}{2}\sum_{n=1}^{\infty} \begin{Bmatrix} (c/R)^n & [(A_n - a_n) \cos n\phi + (B_n - b_n) \sin n\phi] \\ 1 & [A_n \cos n\phi + B_n \sin n\phi] \\ (R/c)^n & [(A_n + a_n) \cos n\phi + (B_n + b_n) \sin n\phi] \end{Bmatrix}, \quad \begin{cases} R > c, & (5.3a) \\ R = c, & (5.3b) \\ R < c. & (5.3c) \end{cases}$$

In (5.2) it would be possible to impose the further condition that u is a solution of Laplace's equation within a circle with radius c and, if so, then separation of variables would give the following relations among the coefficients:

$$A_0 = 0 \tag{5.4a}$$

$$A_n = a_n, \qquad n = 1, 2, \ldots \tag{5.4b}$$

$$B_n = b_n, \qquad n = 1, 2, \ldots . \tag{5.4c}$$

However, greater insight will be obtained if we go further before using the facts which are expressed in (5.4).

From (5.3c) we observe that when the boundary values u and $\partial u/\partial r$ are given (that is when the coefficients a_n, b_n, A_n, and B_n are given) then the value of the function u is determined uniquely within the circle. Now the question becomes: is it possible from (5.3) to determine a unique relation between the boundary values u and $\partial u/\partial r$?—and corresponding between the coefficients?

Case (0): $\bar{\mathbf{r}}'$ *outside C, i.e.* (5.3a)

We obtain a relation among the coefficients when we require that $f(R, \phi) \equiv 0$ for *all* $R > c$, and for $0 \leqslant \phi \leqslant 2\pi$, corresponding to (4.1(0)). This gives

$$0 = -A_0 \ln R \tag{5.5a}$$

$$0 = \tfrac{1}{2}(A_n - a_n), \qquad n = 1, 2, \ldots \tag{5.5b}$$

$$0 = \tfrac{1}{2}(B_n - b_n), \qquad n = 1, 2, \ldots, \tag{5.5c}$$

from which the known relations (5.4) can be derived without difficulties.

Case $(\tfrac{1}{2})+(1)$: $\bar{\mathbf{r}}'$ *on* $C + \bar{\mathbf{r}}$ *inside C, i.e.* (5.3b)+(5.3c)

These two cases, corresponding to (4.1($\tfrac{1}{2}$)) and (4.1(1)), can be considered simultaneously. First the limiting process $R \to c$ is carried out in the result (5.3c), cf. Section 4, Case (1), obtaining

$$\lim_{R \to c} f(R, \phi) = a_0 - A_0 \ln c + \sum_{n=1}^{\infty} \{\tfrac{1}{2}(A_n + a_n) \cos n\phi + \tfrac{1}{2}(B_n + b_n) \sin n\phi\}. \tag{5.6}$$

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The boundary value for u which was inserted in the integral in (5.1) is

$$u(c, \phi) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\phi + b_n \sin n\phi\}. \tag{5.7}$$

Now we require, that the result (5.3b) is equal to $\frac{1}{2}u(c, \phi)$, cf. (5.7), and that the boundary result (5.6) is equal to 1 $u(c, \phi)$, cf. (5.7):

$$(\tfrac{1}{2}): \qquad f(c, \phi) = \tfrac{1}{2}\, u(c, \phi) \tag{5.8a}$$

$$(1): \quad \lim_{R \to c} f(R, \phi) = 1\, u(c, \phi). \tag{5.8b}$$

Comparison of coefficients gives:
($\frac{1}{2}$): i.e. (5.8a):

$$\tfrac{1}{2}a_0 - A_0 \ln c = \tfrac{1}{2}a_0 \tag{5.9a}$$

$$\tfrac{1}{2}A_n = \tfrac{1}{2}a_n, \qquad n = 1, 2, \ldots \tag{5.9b}$$

$$\tfrac{1}{2}B_n = \tfrac{1}{2}b_n, \qquad n = 1, 2, \ldots. \tag{5.9c}$$

(1): i.e. (5.8b):

$$a_0 - A_0 \ln c = a_0 \tag{5.10a}$$

$$\tfrac{1}{2}(A_n + a_n) = a_n, \qquad n = 1, 2, \ldots \tag{5.10b}$$

$$\tfrac{1}{2}(B_n + b_n) = b_n, \qquad n = 1, 2, \ldots. \tag{5.10c}$$

From (5.9b,c) and (5.10b,c) we obtain immediately the relations (5.4b,c) valid for $n = 1, 2, \ldots.$ For $n = 0$ one obtains from both (5.9a) and (5.10a) that

$$A_0 \ln c = 0 \tag{5.11}$$

from which one concludes when $c \neq 1$ that $A_0 = 0$ in accordance with (5.4a), while in the *exceptional case* $c = 1$ it is impossible from (5.11) to conclude $A_0 = 0$.

We see that in general it is possible by means of Green's third identity to create a connection among (the Fourier coefficients of) the boundary values, but for a particular critical magnitude of the boundary curve the connection breaks down for $n = 0$.

6. A Supplementary Condition

The lack of connection between the boundary values in Green's third identity, which we observed in Section 5 for a particular exceptional geometry, can be eliminated by adding a supplementary condition, which we are going to derive here.

The integral which appears in the fundamental formula (4.1) is identically equal to zero when the point $\bar{\mathbf{r}}'$ is placed outside the curve Γ, (4.1(0)). We consider the integral when the point $\bar{\mathbf{r}}'$ is far away from Γ. When the vector $\bar{\mathbf{r}}'$ is expressed as $R\hat{\mathbf{R}}$, where $\hat{\mathbf{R}} = \cos\phi\ \hat{\mathbf{x}} + \sin\phi\ \hat{\mathbf{y}}$ is a unit vector, we want to express the integral in terms of R and ϕ (see Fig. 2). In the integral in (4.1), which is considered as a function $f = f(R, \phi)$, is introduced:

$$G(\bar{\mathbf{r}}', \bar{\mathbf{r}}) = -\frac{1}{2\pi} \ln \rho \tag{6.1a}$$

$$\frac{\partial G}{\partial \nu}(\bar{\mathbf{r}}', \bar{\mathbf{r}}) = \frac{\partial G}{\partial \rho}\frac{\partial \rho}{\partial \nu}, \tag{6.1b}$$

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where

$$\bar{\mathbf{r}}' = R\hat{\mathbf{R}} = R(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) \tag{6.2a}$$
$$\bar{\boldsymbol{\rho}} = \bar{\mathbf{r}}' - \bar{\mathbf{r}} = R\hat{\mathbf{R}} - \bar{\mathbf{r}} \tag{6.2b}$$
$$\rho = |\bar{\boldsymbol{\rho}}| = (R^2 + r^2 - 2\bar{\mathbf{r}}\,R\hat{\mathbf{R}})^{\frac{1}{2}} = R(1+\xi)^{\frac{1}{2}} \tag{6.2c}$$
$$r = |\bar{\mathbf{r}}| \tag{6.2d}$$
$$\xi = (r/R)^2 - 2\bar{\mathbf{r}}\,\hat{\mathbf{R}}/R \tag{6.2e}$$
$$\frac{\partial\rho}{\partial\nu} = -\hat{\mathbf{v}}\cdot\frac{\bar{\boldsymbol{\rho}}}{\rho} = -\hat{\mathbf{v}}\cdot(\hat{\mathbf{R}} - \bar{\mathbf{r}}/R)(1+\xi)^{-\frac{1}{2}}. \tag{6.2f}$$

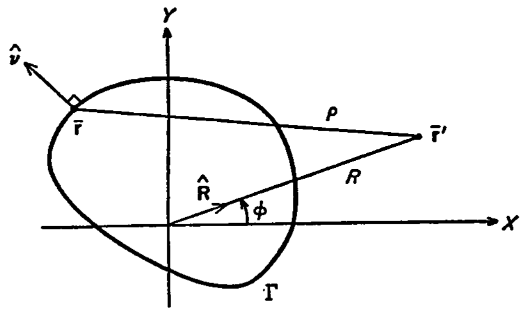

FIG. 2. A plane, closed, smooth, and simple curve Γ with origin placed inside Γ. The point of integration is $\bar{\mathbf{r}}$. The parameter point $\bar{\mathbf{r}}' = R\hat{\mathbf{R}}$ is expressed by means of the unit vector $\hat{\mathbf{R}}$, where $\hat{\mathbf{R}} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$. The distance between $\bar{\mathbf{r}}$ and $\bar{\mathbf{r}}'$ is denoted by ρ.

This gives that the integral in (4.1) is

$$f(R,\phi) = -\frac{1}{2\pi}\int_\Gamma \left\{\frac{\partial u}{\partial\nu}\ln(R(1+\xi)^{\frac{1}{2}}) + u\hat{\mathbf{v}}\cdot(\hat{\mathbf{R}} - \bar{\mathbf{r}}/R)R^{-1}(1+\xi)^{-1}\right\} ds. \tag{6.3}$$

From (6.3) one can derive †

$$f(R,\phi) = -\frac{1}{2\pi}(\ln R)I + O(R^{-1}), \tag{6.4}$$

where

$$I = \int_\Gamma \frac{\partial u}{\partial\nu}\, ds. \tag{6.5}$$

The integral in (6.3) should be zero identically, because it is derived from the integral in (4.1), which is zero identically in the case (4.1(0)). But (6.3) or (6.4) are derived as an expansion in R valid for $R \to \infty$, from which we can only infer that $\lim_{R\to\infty} f(R,\phi) = 0$. This means that we can conclude that the factor integral I, (6.5) necessarily must be equal to zero. Using Gauss' integral theorem the curve-integral (6.5) can be transformed into a plane-integral

$$I = \iint_D \Delta u\, dx\, dy, \tag{6.6}$$

where D denotes the domain within Γ. From (6.4) we have just derived that $I = 0$, cf. (6.5), but this requirement is obviously always satisfied, as is seen from (6.6), because the function u is assumed to be harmonic in D.

† Professor, Dr techn. Erik Hansen has (in 1968) proposed such an expansion in R in connection with a different investigation.

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If in the integral in (6.5) we choose as the integration curve Γ a circle with radius c, where $ds = c\,d\theta$, we get:

$$I = \int_0^{2\pi} \frac{\partial u}{\partial v} c\,d\theta; \tag{6.7}$$

here we insert the Fourier series (5.2) and obtain

$$I = 2\pi A_0. \tag{6.8}$$

As expected, using the known connections (5.4) among the coefficients yields that I in (6.8) is zero. When we investigated the identity (4.1) in connection with a circle with radius c, we found that the known connections (5.4) could be derived, provided that $c \neq 1$. This means that from the identity (4.1) we can conclude that $I = 0$ when $c \neq 1$. But when c had the exceptional value 1, it was impossible to conclude that $A_0 = 0$. We see that this difficulty has influence on the computation of I, (6.8).

In the expansion (6.4) of the integral with $\bar{r}'$ placed outside the curve the integral I, (6.5) was the factor which multiplied $\ln R$. From this fact we now conclude that in the exceptional case, which appears when the curve has the critical magnitude (which in the chosen example of a circle corresponds to the value $c = 1$), then there will appear terms containing the factor $\ln R$. But the presence of such terms can be excluded if it is explicitly required that I must be zero. This requirement we formulate both in the special case where I is given in (6.7), but also in the general form where I is given by (6.5), i.e. we require

$$I = \int_\Gamma \frac{\partial u}{\partial v}\,ds = 0. \tag{6.9}$$

As we have seen it is *necessary only* to add the requirement (6.9) in the *exceptional case*, and, because the requirement (6.9) always has to be satisfied—cf. (6.6), it is possible in all cases to add the requirement (6.9). The requirement (6.9) must not be confused with the condition which one always has to impose on the *known* function in the case of a Neumann boundary value problem. Here in the case of a Dirichlet boundary value problem we arrive at a condition which the *unknown* function must satisfy.

Consequently in connection with integral equations derived from Green's third identity (4.1) it is advantageous always to add the requirement (6.9).

7. Numerical Solution of an Integral Equation together with the Supplementary Condition

Here we show how it can be arranged to combine the solution of the integral equation (2.4) with the requirement (6.9). The practical-numerical procedure can thus be displayed.

When the integral equation (2.4) is considered, where the unknown is $\psi(\bar{r})$, then the supplementary condition (6.9) becomes the following †‡

$$\int_{\Gamma_0 \cup \Gamma_1} \psi(\bar{r})\,ds = 0 \tag{7.1}$$

† In the derivation of (6.9) a simply-connected region was used, while the region in Fig. 1 is doubly-connected. But by means of two closely spaced parallel curves connecting Γ_0 and Γ_1 this domain can be made simply-connected, and (6.9) can be applied. But the sum of the contributions from the parallel curves is zero, and (7.1) is obtained.

‡ Jaswon (1963, p. 28, footnote) has in passing touched upon a similar condition for a more general boundary value problem. A physical interpretation of Jaswon's condition in connection with a practical application of the present investigations (Rasmussen & Christiansen, 1973) was performed at an early stage by Professor, Dr techn. Erik Hansen.

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because $\psi(\bar{r})$ is equal to $(\partial u/\partial v)(\bar{r})$, cf. (2.5). Even though both curves appear in the condition (7.1) it is solely the external curve Γ_0 which determines whether the geometry is critical. See Section 3, (d).

The integral equation (2.4) is replaced by a system of linear algebraic equations by means of a method (Rasmussen & Christiansen, 1973), which is developed from the method of Symm (1963): on each curve, Γ_0 and Γ_1, is picked out an even number of points, namely $2n_0$ and $2n_1$ respectively. These are numbered: $\frac{1}{2}$, 1, $1+\frac{1}{2}$, 2, . . ., $n_l-\frac{1}{2}$, n_l where $l = 0$ or 1. The curves are approximated by straight lines between the points introduced. On each curve Γ_l, where $l = 0$ or 1, the function is assumed to be constant between the point $j-\frac{1}{2}$ and the point $j+\frac{1}{2}$, at the value $\psi_{l,j}$. Now the integrals in (2.4) can be written as a sum, where each term is the product of $\psi_{l,j}$ and the integral of $\ln \rho$ taken over the two straight lines meeting in the point j on curve l. This integral can in all cases be evaluated in closed form (Rasmussen & Christiansen, 1973, Appendix A.3); this result was not given by Symm (1963). Denoting this integral by $M_{k,i;l,j}$ we have indicated that it also depends upon the point $\bar{r}_0$, which is placed on curve Γ_k, in point i, where $i = 1, 2, \ldots, n_k$, and $k = 0$ or 1. Thus the integral equation (2.4) is replaced by n_0+n_1 linear algebraic equations with n_0+n_1 unknowns:

$$\sum_{l=0}^{l=1} \sum_{j=1}^{n_l} M_{k,i;l,j}\psi_{l,j} = \begin{cases} 0; & k = 0, i = 1, 2, \ldots, n_0; \\ 2\pi(u_1-u_0); & k = 1, i = 1, 2, \ldots, n_1. \end{cases} \tag{7.2}$$

The condition (7.1) is similarly transformed into the equation

$$\sum_{l=0}^{l=1} \sum_{j=1}^{n_l} (z_{l,j}^- + z_{l,j}^+)\psi_{l,j} = 0, \tag{7.3}$$

where $z_{l,j}^-$, $z_{l,j}^+$ indicate the length from point j to point $j-\frac{1}{2}$ (reversed) or to point $j+\frac{1}{2}$ respectively; the lengths are measured between the points mentioned—which all are lying on curve Γ_l—along the approximating straight lines. Thus the sum $z_{l,j}^- + z_{l,j}^+$ is an approximation of the arc-length where $\psi_{l,j}$ is assumed to be constant.

The system, (7.2) and (7.3) taken together, with n_0+n_1+1 equations and n_0+n_1 unknowns, is presumably in general incompatible, because it is derived by neglecting some terms which correspond to the quadrature errors. Therefore we solve the combined system using a least square error method (IBM–LLSQ, 1970). This method can also be used if the condition (7.3) is neglected,† and we can then easily carry out the computation both *without* and *with* the condition (7.3), and thus observe the effect of this condition.

8. Numerical Examples

By applying the methods shown in Section 7 to three specific problems we demonstrate by means of the three corresponding numerical examples that the exceptional case is completely eliminated by application of the supplementary condition.

† If the condition (7.3) is left out the system (7.2) can be solved using an ordinary Gaussian elimination (IBM–SIMQ, 1970).

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Example 1

The curves Γ_0 and Γ_1 are two concentric circles C_0 and C_1, with radii c_0 and c_1, respectively ($c_0 > c_1$). This problem is considered in Section 3. The unique (two dimensional) rotationally symmetric potential is

$$u = u_0 + \frac{\ln(r/c_0)}{\ln(c_1/c_0)}(u_1 - u_0); \qquad c_1 \leqslant r \leqslant c_0,$$

from which we derive the exact constant solutions

$$\psi_0 = -\left.\frac{\partial u}{\partial r}\right|_{r=c_0} = -\frac{u_1 - u_0}{c_0 \ln(c_1/c_0)}, \tag{8.1a}$$

$$\psi_1 = +\left.\frac{\partial u}{\partial r}\right|_{r=c_1} = +\frac{u_1 - u_0}{c_1 \ln(c_1/c_0)}. \tag{8.1b}$$

Notice that the solutions (8.1) satisfy the two linear algebraic equations (3.1). The exceptional case in the sense of Section 3 occurs when the exterior radius c_0 is equal to one; cf. (3.2a). In the following we consider the case $c_0 = 2c_1$. The equations (7.2) either *without* or *with* the condition (7.3) are solved, when c_0 takes on the values 0·1, 1·0 and 10·0, for different values of $N = n_0 + n_1$, when $n_0 = n_1$. The relative errors are shown in Table 1.

TABLE 1

Geometry: two concentric circles with radii c_0 and c_1 ($c_0 = 2c_1$). The relative error = (approximation − true)/true, given in units of 10^{-3}, for ψ_0 and ψ_1 (8.1). $N = n_0 + n_1$ is the number of unknown function values. −, + denotes without, with the supplementary condition (7.3)

N	− / +	$c_1 = 0·05$	$c_0 = 0·1$	$c_1 = 0·5$	$c_0 = 1·0$ (exceptional)	$c_1 = 5·0$	$c_0 = 10·0$
12	−	13·5	8·8	−4·0	−999·8	13·6	18·4
	+	13·6	8·9	29·2	26·3	13·7	18·5
24	−	3·1	1·9	−1·1	−999·6	3·1	4·4
	+	3·1	1·9	7·1	6·7	3·1	4·4
48	−	0·8	0·4	−0·3	−998·7	1·0	1·1
	+	0·8	0·4	1·8	1·7	1·1	1·0

From Table 1 we derive the following findings:

(a) at the exceptional case the results obtained *without* the supplementary condition are grossly wrong, while *with* the condition the results are accurate;

(b) away from the exceptional case there is not much difference between the results obtained *without* or *with* the supplementary condition;

(c) the error of the results obtained *with* the supplementary condition decreases approximately as h^2 when $N \simeq h^{-1}$ increases;

(d) the grossly wrong results in the exceptional case appear on the *exterior* curve corresponding with the fact that ψ_0 cannot be determined uniquely from (3.1) when $c_0 = 1$.

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Example 2

Analogous with Example 1; i.e. with $c_0 = 2c_1$. Here is considered the case shown at the bottom of Table 1, i.e. the value $N = n_0+n_1$ is kept fixed equal to 48. The relative error for the solution ψ_0 on the *exterior* circle, cf. (d) above, is found both *without*† and *with* the condition (7.3). In Fig. 3 the results are displayed as two functions of c_0, where $0{\cdot}5 \leqslant c_0 \leqslant 1{\cdot}0$.

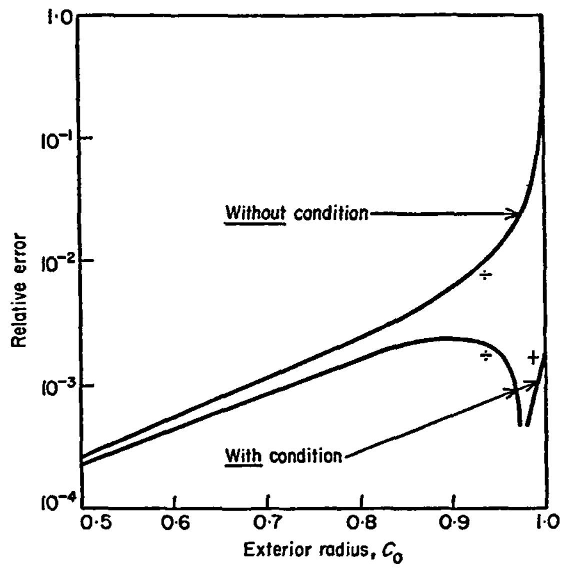

FIG. 3. Geometry: two concentric circles with radii c_0 and c_1, where $c_0 = 2c_1$. The exceptional geometry occurs when $c_0 = 1$. The relative error = (approximation−true)/true, for the solution ψ_0, cf. (8.1a), of the exterior circle, determined both *without* and *with* the supplementary condition (7.3). The errors are here presented as functions of c_0, where $0{\cdot}5 < c_0 < 1{\cdot}0$. The signs +, ÷ denote that the relative error is positive, negative.

From Fig. 3 we derive the following findings: (i) and (ii) see the findings (a) and (b) at Table 1; (iii) also when the geometry is different from, but somewhat close to, the critical the results obtained *without* the supplementary condition are inaccurate.

Example 3

The curves Γ_0 and Γ_1 are two confocal ellipses:

$$\Gamma_0: \quad x = a_0 \cos\theta,\ y = b_0 \sin\theta; \qquad 0 \leqslant \theta \leqslant 2\pi \tag{8.2a}$$

$$\Gamma_1: \quad x = a_1 \cos\theta,\ y = b_1 \sin\theta; \qquad 0 \leqslant \theta \leqslant 2\pi \tag{8.2b}$$

with

$$a_0 = c\cosh\mu_0, \qquad b_0 = c\sinh\mu_0 \tag{8.2c}$$

$$a_1 = c\cosh\mu_1, \qquad b_1 = c\sinh\mu_1, \tag{8.2d}$$

where $0 < \mu_1 < \mu_0$, while c is a constant. Using Moon & Spencer (1961, pp. 17–19) one can find as the unique (two dimensional) solution of the potential problem:

$$u = u_0 + \frac{\mu_0-\mu}{\mu_0-\mu_1}(u_1-u_0); \qquad \mu_1 \leqslant \mu \leqslant \mu_0,$$

† The case without condition has been completed using (IBM–SIMQ, 1970) and not (IBM-LLSQ, 1970), cf. Section 7, footnote †. This does not change the results essentially.

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from which one derives the exact solutions:

$$\psi_0 = \left.\frac{\partial u}{\partial v}\right|_{\mu=\mu_0} = \frac{u_1-u_0}{\mu_0-\mu_1} c^{-1}(\cosh^2\mu_0-\cos^2\theta)^{-\frac{1}{2}}, \tag{8.3a}$$

$$\psi_1 = \left.\frac{\partial u}{\partial v}\right|_{\mu=\mu_1} = -\frac{u_1-u_0}{\mu_0-\mu_1} c^{-1}(\cosh^2\mu_1-\cos^2\theta)^{-\frac{1}{2}}. \tag{8.3b}$$

In the following we consider only the case: $\mu_1 = 1, \mu_0 = 2$. When the exterior curve is an ellipse with semiaxes a_0 and b_0 the exceptional case in the sense of Section 3 occurs when $a_0+b_0 = 2$; cf. (3.2b). When $\mu_0 = 2$ this corresponds with $c = 0{\cdot}2707\ldots$; cf. (8.2c). The equations (7.2) either *without* or *with* the condition (7.3) are solved, when $n_0 = n_1$. The relative error on ψ_0 (the solution on the exterior curve Γ_0) is nearly the same for all points on Γ_0, while the relative error on ψ_1 (the solution on the interior curve Γ_1) is *not* the same for different points on Γ_1: the largest (and absolutely largest) relative error occurs at the end-points of the large semiaxes.† Thus the relative error at the end-points of the large semiaxes is considered to be representative for the relative errors on the two curves. In Table 2 this error is shown for $c = 0{\cdot}1$, $0{\cdot}2707\ldots$ and $1{\cdot}0$ for different values of $N = n_0+n_1$, when $n_0 = n_1$.

TABLE 2

Geometry: two ellipses (8.2). *The relative error* = *(approximation — true)/true at the end-points of the large semiaxes given in units of* 10^{-3}, *for* ψ_0 *and* ψ_1 (8.3). $N = n_0+n_1$ *is the number of unknown function values.* —, + *denote without, with the supplementary condition* (7.3)

N	— / +	$c = 0{\cdot}1$		$c = 0{\cdot}2707\ldots$ (critical)		$c = 1.0$	
		$\mu_1 = 1$	$\mu_0 = 2$	$\mu_1 = 1$	$\mu_0 = 2$	$\mu_1 = 1$	$\mu_0 = 2$
12	—	19·3	0·3	7·2	−1000·3	19·5	19·8
	+	19·7	1·2	30·3	19·2	19·7	19·9
24	—	9·2	0·3	6·2	−1000·0	9·2	5·2
	+	9·3	0·4	12·0	5·5	9·2	5·2
48	—	3·0	0·1	2·2	−998·3	2·9	1·4
	+	3·0	0·1	3·6	1·4	2·9	1·4

From Table 2 we derive the following findings: (i) and (ii) see the findings (a) and (b) at Table 1; (iii) the error of the results obtained with the supplementary condition decreases faster than h^1 when $N \simeq h^{-1}$ increases; (iv) the grossly wrong results in the exceptional case appears on the *exterior* curve, cf. finding (d) at Table 1.

9. Discussion

By means of Green's third identity one can derive an integral equation for the first boundary value problem in two space-dimensions for an harmonic function. Even

† The relative error of the solution at a point of a curve seems to be positively correlated with the curvature (= radius^{-1}) at this point: the exterior curve Γ_0 is more nearly circular giving nearly the same error, while the interior curve Γ_1 deviates somewhat more from a circle having the absolutely largest relative error where the curvature is largest.

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though the problem has a unique solution it turns out that in a particular exceptional case when the geometry has a particular critical magnitude the integral equation does not have a unique solution. Therefore we have investigated Green's third identity in order to find whether this identity always exhibits a close connection between the boundary values u and $\partial u/\partial v$ of an harmonic function. This is normally the case, but we can very simply display the exceptional case which occurs when the geometry of the problem has a particular critical magnitude. We eliminate this exceptional case by adding a supplementary integral-condition (6.9), which would be automatically satisfied, the exception being only for a particular valued geometrical boundary, where it may fail to be fulfilled.

The value of this condition is demonstrated by three examples: the linear algebraic equations (7.2), which correspond with the integral equation (2.4), are solved both *without* and *with* the supplementary condition (7.3), which corresponds with (6.9). Tables 1 and 2 show that the application of the condition in the exceptional case is quite decisive, while causing no harm nor difficulty in all other cases.

The analytical investigation shows that the exceptional case, where the integral equations do not have a unique solution, does occur for a certain particular geometry. But from a computational point of view it turns out also when the geometry is different from but somewhat close to the critical, it is difficult to obtain an accurate solution. This effect where the analytical difficulty is smeared out in the practical case is illustrated in Fig. 3, showing that the solutions are less accurate in a fairly large zone around the theoretical critical geometry. This makes it even more essential to identify and eliminate the exceptional case.

We conclude that adding the condition (6.9) is essential, thus, quite simply, eliminating the exceptional case which can occur in connection with integral equations for two dimensional harmonic problems.

The impulse to the present investigation stems from the following two works.

(1) A Master-Thesis (Pedersen, 1971) on biharmonic functions and elastic plates; Peter Hougård Pedersen is thanked for long and detailed discussions, and for reading a preliminary Danish version of the manuscript.

(2) A computation of a two-dimensional potential problem in connection with electrochemical machining problems (Rasmussen & Christiansen, 1973); Henning Rasmussen is thanked for discussions.

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Appendix

Some definite integrals

The following integrals can be derived from (Gradshteyn & Ryzhik, 1965). The numbers of the formulas used are written in [ ].

$$0 \leqslant x \leqslant 2\pi, 0 < c, 0 \leqslant R; \qquad \rho^2 = c^2 + R^2 - 2cR \cos x.$$

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$$\frac{c}{2\pi}\int_0^{2\pi}(c-R\cos x)\rho^{-2}\,dx = \begin{bmatrix}1, & R<c\\ \frac{1}{2}, & R=c\\ 0, & R>c\end{bmatrix}; \quad [3.613-2] \tag{A1}$$

$$\frac{c}{2\pi}\int_0^{2\pi}(c-R\cos x)\rho^{-2}\cos nx\,dx = \frac{1}{2}\cdot\begin{bmatrix}+(R/c)^n, & R<c\\ 0, & R=c\\ -(c/R)^n, & R>c\end{bmatrix},$$

$$n = 1, 2, \ldots; \quad [3.613-2]\dagger \tag{A2}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\ln\rho\,dx = \begin{pmatrix}\ln c, & R\leqslant c\\ \ln R, & R\geqslant c\end{pmatrix}; \quad [4.224-14] \tag{A3}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\ln\rho\cos nx\,dx = -\frac{1}{2n}\cdot\begin{pmatrix}(R/c)^n, & R\leqslant c\\ (c/R)^n, & R\geqslant c\end{pmatrix}, n = 1, 2, \ldots. \quad [4.397-6] \tag{A4}$$

† And application of $\cos x \cos nx = \frac{1}{2}[\cos(n+1)x+\cos(n-1)x]$.

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