CHIEF

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A NUMERICAL STUDY OF THE COEFFICIENT MATRIX OF THE BOUNDARY ELEMENT METHOD NEAR CHARACTERISTIC FREQUENCIES

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In this paper the well-known failure in numerical computations of the exterior surface Helmholtz integral equation at certain characteristic frequencies is investigated. The problem of characteristic frequencies is numerically reflected in an ill-conditioning of the coefficient matrix arising from the transformation of the integral equation to a linear set of equations. By means of a rank revealing factorization the rank deficiency of the coefficient matrix is determined, and thereby the necessary number of equations to be added to the original system of equations can be found. One of the most popular methods to overcome this "non-uniqueness" problem is the Combined Helmholtz Integral Equation Formulation (CHIEF) proposed by Schenck [1]. However, CHIEF points placed on or near a nodal surface of the corresponding interior problem do not provide a linearly independent constraint, and the problem of selecting "good" CHIEF points is still a topic of considerable interest. In this paper the CHIEF is used with a rank revealing factorization, and it is shown that the necessary number of "good" CHIEF points can be predicted; furthermore, a method of determining whether a CHIEF point is "good" is presented. The new approach has been tested on an axisymmetric BEM formulation with one or several CHIEF points.

characteristic

1. INTRODUCTION

Boundary element methods (BEM) have successfully been used for solving radiation and scattering problems in acoustics for some years. One of the most significant advantages of BEM compared to the finite element method (FEM) is that a three-dimensional problem may be described by a two-dimensional integral equation so that only the boundary of the (e.g., exterior) domain has to be discretized. Not only does this solve the problem of handling domains of infinite extent, which obviously are difficult to handle with FEM, but the work of discretizing a problem to obtain a numerical solution is significantly reduced.

One of the problems frequently addressed in BEM is the problem of characteristic frequencies in exterior boundary integral formulations. These characteristic frequencies are a result of the formulation into an integral equation (Fredholm integral equation of the second kind), and are the eigenfrequencies of a corresponding interior problem, but they have no physical meaning for the exterior problem under consideration.

This "non-uniqueness" problem is numerically manifested in a rank deficiency of the BEM coefficient matrix, and in order to obtain the unique solution that is known to exist

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analytically, several modified integral equation formulations that provide additional constraints to the original system of equations have been proposed [1–13]. A summary of the different formulations and their advantages/disadvantages has recently been given in reference [2]. Suffice it to say here that in general a theoretically robust formulation suffers from being complicated and/or computationally inefficient. On the other hand, a simple formulation, easy to implement, like the CHIEF proposed by Schenck [1], leaves the user without the assurance of having obtained the correct solution.

In the CHIEF formulation proposed by Schenck [1] the Helmholtz integral equation for exterior problems is used with interior points (CHIEF points) to produce the constraint that is necessary to obtain a unique solution, when the constraint is satisfied along with the surface Helmholtz integral equation formulation. This formulation has the drawback that points placed on a nodal surface of the corresponding interior problem do not provide a linearly independent constraint and are therefore useless ("bad" CHIEF points). The term "good" CHIEF points is used for points that do provide a linearly independent constraint. Another problem in using CHIEF is how to determine how many "good" CHIEF points are needed to obtain the correct solution: recently it has been reported [2] that the use of only one "good" CHIEF point is not in general sufficient at higher characteristic frequencies. In reference [2] it was suggested that this phenomenon was due to a rank deficiency greater than one at higher characteristic frequencies.

The problem of characteristic frequencies and interior nodal surfaces is of practical importance due to the numerical treatment. When discretizing the problem bad solutions occur not only at the characteristic frequencies, but in a range of frequencies near the characteristic frequencies. Likewise are CHIEF points placed near (and not only on) the interior nodal surfaces' "bad" CHIEF points. The "bandwidth" of the zone leading to false solutions depends on the frequency, the sophistication of the method (the order of the polynomials used to approximate the geometry and the acoustic variables), and on the "fineness" of the mesh used. Since it is unlikely that one would choose a frequency exactly equal to a characteristic frequency, it should always (at least in theory) be possible to circumvent the non-uniqueness problem by making the mesh finer. However, this is a strategy that leads to an enormous amount of computational work and storage required. According to this point of view the use of CHIEF and other methods to circumvent non-uniqueness may be regarded as methods to enable the user to maintain a mesh as coarse as possible for a given accuracy. (A rule of thumb is to choose the mesh size to be half or one-third of a wavelength.) The computational work of a formulation to circumvent non-uniqueness is therefore an important parameter to be considered.

Recently, singular value decomposition (SVD) has been used to evaluate some of the new methods of circumventing the non-uniqueness problem [14]. In this paper the SVD is used not only to detect non-uniqueness but also to estimate the quality of the CHIEF point, including problems where a rank deficiency greater than one occurs.

2. FORMULATION

For time-harmonic waves, and with the time factor $e^{i\omega t}$ omitted, the general Helmholtz integral formula [15] can be expressed in terms of the complex pressure p and the complex surface velocity normal to the body v:

$$C(P)p(P)=\int_S\left(p(Q)\frac{\partial G(R)}{\partial n}+\mathrm{i}kz_0 v(Q)G(R)\right)\mathrm{d}S+4\pi p^I(P). \qquad (1)$$

This formula is valid in an infinite homogeneous medium (e.g., air) outside a closed body B with a surface S. In the medium p satisfies $\nabla^2 p + k^2 p = 0$. Q is a point on the surface S, and P is a point either inside, on the surface of, or outside the body B. The quantity $R = |P - Q|$ is the distance between P and Q, and $G(R) = e^{-ikR}/R$ is the free-space Green function, $k = \omega/c$ is the wavenumber, where ω is the circular frequency and c is the speed of sound, i is the imaginary unit, and z_0 is the characteristic impedance of the medium and n is the unit normal to the surface S at the point Q directed away from the body. The quantity $C(P)$ has the value 0 for P inside B and 4π for P outside B. In the case of P on the surface S, $C(P)$ equals the solid angle measured from the medium ($= 2\pi$ for a smooth surface) [10]. Equation (1) may be solved numerically by defining a mesh to discretize the body B. The acoustic variables p and v are then supposed to follow a specific shape (e.g., quadratic) between the nodes of the mesh. In this way the geometry and the acoustic variables of the problem are defined by the values on the finite number (M) of nodes. In many problems the values of v are known or may be expressed in terms of p by an impedance relation: $p(Q) = z(Q)v(Q)$ (note, however, that this formula is useful only for a locally reacting surface). In order to obtain M equations matching the M unknown values of the pressure p, the point P is placed on the M nodes of the surface S. The resulting equations may then be expressed in matrix form as

$Dx =$

$$\mathbf{D}\mathbf{p} = \mathbf{M}\mathbf{v} + \mathbf{p}^{I}, \tag{2}$$

where bold capital letters denote matrices and bold lower case letters denote vectors. The use of **M** and **D** for the matrices refers to the fact that they contain integrals over the Green function and its derivative, respectively—these terms are often interpreted as the monopole term and the dipole term.

Using the boundary conditions with equation (2) reduces equation (2) to

$$\mathbf{C}\mathbf{x} = \mathbf{y}, \tag{3}$$

where **x** is the unknown vector and **y** is the known vector. For the problem of scattering from a rigid surface **C** equals **D** and **y** equals $\mathbf{p}^{I}$, and for a radiation problem where **v** is known, **C** equals **D** and **y** equals **Mv**.

$v = \frac{\partial p}{\partial n}$

3. SINGULAR VALUE DECOMPOSITION

By the transformation of the integral equation to a linear set of equations the problem of characteristic frequencies becomes reflected in an ill-conditioning of the coefficient matrix **C**. The condition number κ may roughly be described as the factor a disturbance of an element in the matrix **C**, or the right side **y**, may be multiplied by in the solution vector **x**, and the matrix is ill-conditioned if the condition number is large. As the elements of **C** are a result of approximations (discretization and numerical integration) the uncertainty of these elements is usually much larger than the machine epsilon (the accuracy with which numbers are represented internally in the computer). If the condition number is infinite the matrix is singular.

In handling singular/ill-conditioned matrices singular value decomposition (SVD) is often considered the ultimate tool (see, e.g., the book by Press *et al.* [16]). The singular value decomposition of a square $N \times N$ matrix **A** is defined as

$$\mathbf{A} = \mathbf{U}\mathbf{W}\mathbf{V}^{T}, \tag{4}$$

where $\mathbf{V}^T$ denotes the transpose of the matrix **V**. This decomposition is always possible [16], and programs to perform the SVD are available both for mainframes and for

PCs (routines are listed, e.g., by Press *et al.* [16]). The matrices **U** and **V** are orthogonal, i.e.,

$$\sum_{i=1}^{N} u_{ik} u_{in} = \delta_{kn} \begin{Bmatrix} 1 \leqslant k \leqslant N \\ 1 \leqslant n \leqslant N \end{Bmatrix}, \qquad \sum_{j=1}^{N} v_{jk} v_{jn} = \delta_{kn} \begin{Bmatrix} 1 \leqslant k \leqslant N \\ 1 \leqslant n \leqslant N \end{Bmatrix}, \tag{5a, b}$$

where u_{ik} denotes the element in row i and column k, and δ is the Kronecker delta. **W** is a diagonal matrix, and the values w_{jj} (or in brief w_j) in the diagonal of **W** are called the singular values. Without loss of generality the columns of the matrices **U**, **V** and **W** may be arranged in order of descending w_j's so that w_1 is the largest element and w_N is the smallest. Since **U** and **V** are orthogonal their inverses equal their transposes, and the inverse of **A** is

$$\mathbf{A}^{-1} = \mathbf{V} \cdot [\mathrm{diag}(1/w_j)] \cdot \mathbf{U}^{\mathrm{T}}. \tag{6}$$

Analytically this formula behaves well if none of the w_j's are zero, but numerical problems arise if one or several of the w_j's are small compared to the accuracy of the elements of **A**. The condition number κ of a matrix is defined as the ratio w_1/w_N, and, as previously mentioned, the matrix is said to be ill-conditioned/singular if this ratio is large/infinite.

In order to investigate the properties of the SVD further it is convenient to regard **A** as the matrix of a linear mapping,

$$\mathbf{y} = \mathbf{A}\mathbf{x}: \tag{7}$$

i.e., the vector **x** is mapped on to the vector **y** by equation (7). The columns in **U** and **V** calculated by a SVD are connected by the simple relation

$$\mathbf{A}\mathbf{v}_j = w_j \mathbf{u}_j. \tag{8}$$

Any vector $\mathbf{x} \in \mathbb{R}^N$ may be expressed by the columns of **V**,

$$\mathbf{x} = \xi_1 \mathbf{v}_1 + \xi_2 \mathbf{v}_2 + \cdots + \xi_N \mathbf{v}_N, \tag{9}$$

and the vector **y** on to which **x** is mapped by equation (7) may be expressed by using equation (8) and equation (9) as

$$\mathbf{y} = \sum_{j=1}^{N} \xi_j w_j \mathbf{u}_j. \tag{10}$$

In this way (in view of equation (8)) the w_j's may be regarded as the magnification of the $\mathbf{v}_j$'s when mapped on to the corresponding $\mathbf{u}_j$'s (in some sense similar to a "transfer function"). If **A** is regular (non-singular) then when **x** goes through all possible combinations of the columns of **V** (by equation (9)) **y** will go through all possible combinations of the columns of **U**. Consequently, the columns in **V** span an orthogonal basis for the solution space of **A**, and the columns of **U** span an orthogonal basis for the range of **A**. (Range refers to "what may be 'reached' by the linear mapping defined by **A**".)

If **A** is singular then one or several of the w_j's are zero (say, the last $N - R$ ones, $R < N$) and the corresponding last column(s) of **V** are called singular vectors and are by equation (8) mapped into the zero-vector:

$$\mathbf{A}\mathbf{v}_j = \mathbf{0}. \tag{11}$$

In this case **A** is said to be rank deficient (the rank of **A** is R), and two additional subspaces are needed in the discussion of the mapping. The last $N - R$ columns of **V** are called the null space of **A** (since they are mapped into the zero vector), and the corresponding (last $N - R$) columns of **U** are called the orthogonal complement of **A** (since this vector space

TABLE 1

The connection between the four fundamental subspaces and the SVD

Name	Basis vectors	Dimensions
Range	$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_R$	R
Orthogonal complement	$\mathbf{u}_{R+1}, \ldots, \mathbf{u}_N$	$N-R$
Solution space	$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_R$	R
Null space	$\mathbf{v}_{R+1}, \ldots, \mathbf{v}_N$	$N-R$

may not be "reached" by **A**). The solution space of **A** is then spanned by the first R columns of **V**, and the range of **A** by the first R columns of U. These properties are summarized in Table 1.

If **A** is the coefficient matrix of a system of equations to be solved for a known right side **b**, i.e.,

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{12}$$

a singular matrix corresponds to one of two alternatives: either the system of equations has no solution (**b** is not in the range of **A**); or the system of equations has one or several infinities of solutions, since in this case any combination of the zero-vectors ($\mathbf{v}_{R+1}, \ldots, \mathbf{v}_N$) may be added to a specific solution (**b** is in the range of **A**, and may be expressed as a linear combination of the first R columns of **A**). In contrast to, e.g., the simple source formulation [1], a solution is known to exist at the characteristic frequencies when using the Helmholtz integral equation, and the latter alternative is therefore the actual one. The number of zero w_j elements is, as previously stated, the rank deficiency of the matrix **A**, and is also the number of missing linearly independent equations that must be added in order to maintain a system in which the number of linearly independent equations equals the number of unknowns. The problem is therefore to add additional constraints to the system of equations in order to obtain a unique solution (or in other words in order to pick out the correct combination of the singular vector(s)).

Numerically, an exact singular matrix seldom occurs, but the situation described above is manifested in an ill-conditioned matrix. The numerical rank of a matrix may be defined as the number of w_j's below a certain value. If equation (6) is used without modifications at a characteristic frequency, the solution vector may be drawn towards infinity in a direction that is almost a singular vector or, in the case of a rank deficiency higher than one, a combination of the singular vectors due to approximations made and/or round-off errors. (The solution is polluted with a constant times the singular vector(s).)

4. USING SVD IN BEM

In the CHIEF approach one uses the Helmholtz integral equation with interior points in order to produce the necessary constraints at characteristic frequencies [1]. However, a CHIEF point placed on or near a nodal surface of the corresponding interior problem does not provide a linearly independent constraint and is useless [11]. One approach to this problem is to distribute a number of CHIEF points hoping that a sufficient number of CHIEF points do not fall on or near a nodal surface. The resulting overdetermined system of equations is then solved by means of a least-squares procedure. Note that the SVD may also be used in the case of an $M \times N$ matrix ($M > N$). In this case **U** is a $M \times N$ column-orthogonal matrix, and the matrices **V** and **W** are both $N \times N$. The (generalized) condition number is still defined as the ratio w_1/w_N. A matrix for which the number of rows does not equal the number of columns is termed a rectangular matrix, whereas a matrix for which the number of rows equals the number of columns is called square. In terms of accuracy the SVD is more favourable than the normal least-squares procedure for $\mathbf{A}^T\mathbf{A}$ since the matrix $\mathbf{A}^T\mathbf{A}$ has the condition number κ^2 if the rectangular matrix $\mathbf{A}$ has the condition number κ.

The theory in the last paragraph was discussed for the case of a real matrix $\mathbf{A}$. Handling the complex BEM coefficient matrix in equation (3) may be done either by a complex SVD routine or by rewriting the complex system of equations in equation (3) into a real system of equations. With $\mathbf{C} = \mathbf{A} + i\mathbf{B}$, $\mathbf{x} = \mathbf{x}^R + i\mathbf{x}^I$, and $\mathbf{y} = \mathbf{y}^R + i\mathbf{y}^I$ one may rewrite equation (3) as

$$\underbrace{\begin{pmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix}}\begin{pmatrix} \mathbf{x}^R \\ \mathbf{x}^I \end{pmatrix} = \begin{pmatrix} \mathbf{y}^R \\ \mathbf{y}^I \end{pmatrix}. \tag{13}$$

If $\mathbf{x}_0 = \mathbf{x}_0^R + i\mathbf{x}_0^I$ is a singular vector,

$$\begin{pmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix}\begin{pmatrix} \mathbf{x}_0^R \\ \mathbf{x}_0^I \end{pmatrix} = \mathbf{0}, \tag{14}$$

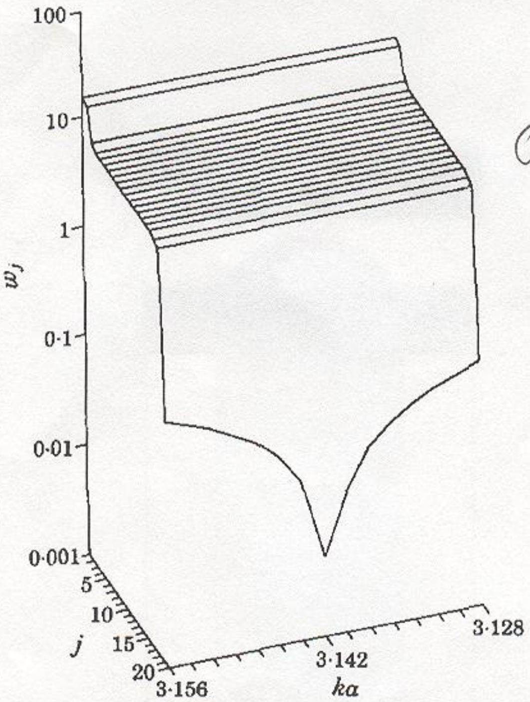
then it immediately follows that $-\mathbf{x}_0^I + i\mathbf{x}_0^R$ is a singular vector as well, and this vector is evidently orthogonal to $\mathbf{x}_0 = \mathbf{x}_0^R + i\mathbf{x}_0^I$. It can be shown that the singular values of the matrix in equation (12) are always pairs of same value due to the special structure of the $2N \times 2N$ matrix, and that the two columns in $\mathbf{U}$ and $\mathbf{V}$ corresponding to the two identical w_j's have the above-mentioned property. In the following examples only one value of the pair of w_j's is shown. In order to investigate the behaviour of the singular values (the w_j's) near a fictitious eigenfrequency, SVD has been performed on the square BEM coefficient matrix in the case of a rigid sphere. The axisymmetry of the geometries in the examples presented has been used, and hence the test case is made on an axisymmetric BEM formulation where only the generator of the bodies has been discretized. The generator of the sphere was divided into 19 line segments and 20 nodes, and p was assumed to follow a linear variation between the nodal values (v is zero).

One of the important properties of the SVD is illustrated by Figure 1, which shows the singular values of the BEM coefficient matrix in a range of frequencies near the first characteristic frequency at $ka = \pi$. From the figure it is evident that the first 19 singular values are practically constant in the range of $ka = 3{\cdot}128$ to $ka = 3{\cdot}156$, whereas the last singular value shows a strong dependence of the "distance" to the characteristic frequency. Since the first singular value w_1 is almost constant in this range of frequencies, the condition number $\kappa = w_1/w_N$ ($N = 20$) has inverse dependence on the last singular value and becomes large as the frequency approaches the characteristic frequency. Hence the problem of characteristic frequencies is directly reflected in the last singular value. Since only one singular value becomes small at $ka = \pi$, only one good CHIEF point is needed to add sufficient constraint to the system of equations, and the condition number calculated by the SVD for the overdetermined system of equations produced by the BEM coefficient matrix with a CHIEF point in the centre of the sphere is $\kappa = 2{\cdot}6$, which is very close to the best possible theoretical value, unity.

5. NUMERICAL RANK

As briefly mentioned in section 3, one must decide on a threshold for the w_j's under which the matrix is said to be (numerically) rank deficient. In order to investigate the connection between the last (smallest) singular value and the error made by the BEM formulation without any CHIEF points, the error made by the BEM in the case of

3-D plot



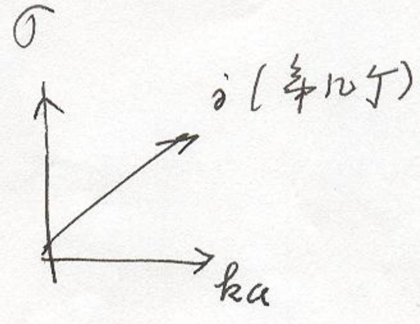


Figure 1. The 20 singular values for a 20-node discretization of a rigid sphere near first characteristic frequency at $ka = \pi$.

scattering of a plane wave of magnitude unity (dimensionless for convenience) by a rigid sphere, and the last singular value, are plotted as functions of the dimensionless wavenumber ka in Figure 2. The error is calculated as the length of the residual vector—the residual vector is the vector containing the difference between the analytical solution and the BEM solution at the nodes. For this figure two BEM calculations have been made: the first with a 20-node discretization (the same as used for Figure 1) and the second with a 40-node discretization.

The figure shows that for both discretizations the last singular values are practically identical whereas the error depends strongly on the discretization. This may be

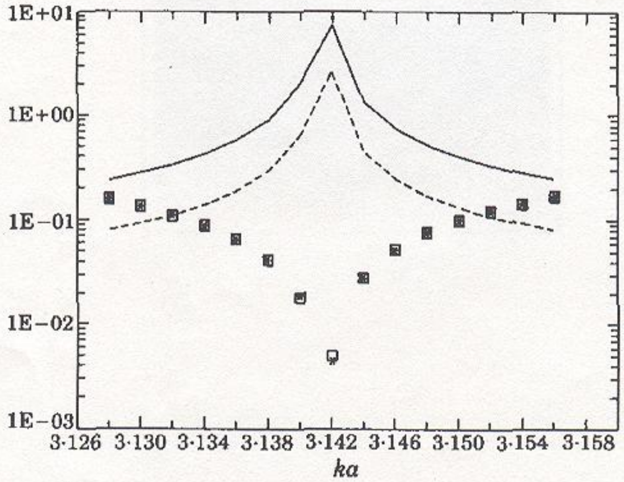


Figure 2. Solution error and last singular value near first characteristic frequency at $ka = \pi$. □, Last singular value for 20-node discretization; $*$, last singular value for 40-node discretization; ——, solution error for 20-node discretization; ----, solution error for 40-node discretization.

explained by the close relation between the last singular value and the condition number: as mentioned previously the condition number may be regarded as the "blow-up" factor for the error made (due to the approximations) by the BEM formulation, and evidently the degree of approximation is larger for the 20-node formulation than for the 40-node formulation. Hence, although the last singular value is the same in the two situations, the resulting error is not. However, the last singular value obviously proves to be a good estimate of the error, and thus it is possible to use the last singular value as an error indicator. Hence it is possible to decide on a threshold for the singular values, and to decide to improve the standard BEM formulation, e.g., by adding CHIEF points if this threshold is crossed. As mentioned above, the resulting error is not a function of the last singular value only, and hence the threshold should be chosen with regard to the actual implementation of the Helmholtz integral equation. For any practical implementation one could keep the ratio of wavelength to element size constant, and examine the error made by this particular implementation for cases in which the non-uniqueness problem does not occur. The threshold could then be defined by examining the last singular value for the implementation as the frequency is moved close to a characteristic frequency. The threshold would then be the largest of the last singular values in the band of frequencies where the error is unacceptably high compared to the level found in the first experiment.

6. ADDING A CHIEF POINT

Once one has determined the number of good CHIEF points needed to pick out the correct solution to the problem by inspecting the singular values of the BEM coefficient matrix, it becomes important to be able to estimate the quality of a CHIEF point. Note that if the complex system of equations has been translated to a real system by equation (13), a CHIEF point provides two independent equations to be satisfied along with the normal BEM coefficient matrix corresponding to the two singular vectors shown to exist in section 4 for the two identical singular values.

The SVD provides a very good tool for deciding whether a CHIEF point is good: the singular vectors. When a matrix $\mathbf{A}$ is rank-one deficient (i.e, when the last singular value is zero), any constant times the singular vector may be added to a specific solution without altering the right side. Consider the system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{y}\ (= \mathbf{A}(\mathbf{x} + t\mathbf{x}_0)), \qquad t \in \mathbb{R}. \tag{15}$$

In order to pick out a solution from this infinity of solutions one may add an extra equation (a CHIEF point) to lay the necessary constraint on the parameter t. This results in a rectangular system of equations. If the extra equation adds the necessary constraint (a "good" CHIEF point) then the rectangular system is non-singular, implying that the singular vector $\mathbf{x}_0$ of equation (15) is not a singular vector for the rectangular system

nontrivial mode

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{a}_{ex}^{\mathrm{T}} \end{pmatrix} (\mathbf{x}_0) \neq \mathbf{0} \quad \Leftrightarrow \quad \mathbf{a}_{ex}^{\mathrm{T}} \cdot \mathbf{x}_0 \neq 0, \tag{16a, b}$$

since $\mathbf{A}\mathbf{x}_0 = \mathbf{0}$. If equation (16b) is true (in practice the left side must be greater than a certain threshold) then the rectangular system of equations is non-singular, and the solution may be found as the least-squares solution of the rectangular system.

The left side of equation (16b) may be used as a quality control of the extra equation, since a small product implies that no additional constraint has been obtained.

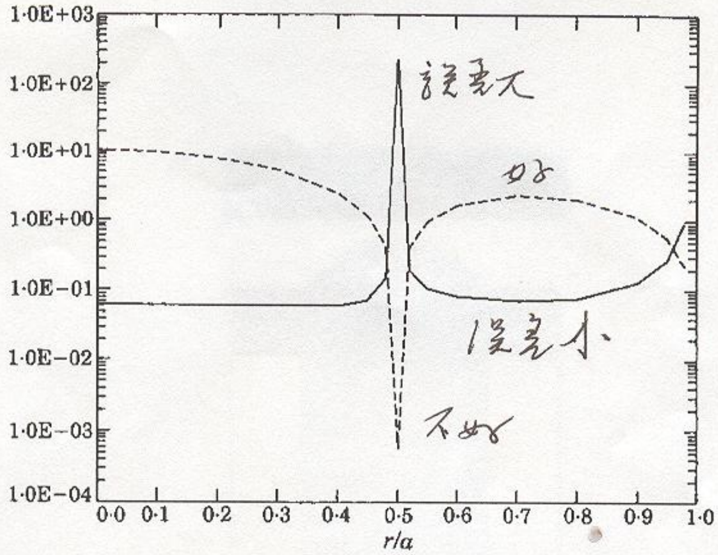


Figure 3. Quality control and solution error as functions of the ρ co-ordinate ($\theta, z = 0$) of the CHIEF point for the case of a rigid sphere at $ka = 6{\cdot}2832$. ——, Solution error; ----, quality control.

The theory described above is also valid for the case of the $2N \times 2N$ real system translated from the complex BEM coefficient matrix. Here the combination of two singular vectors corresponding to two identical singular values is to be found from the two extra equations added by a CHIEF point. Due to the special symmetry of the equations, the largest of the dot products of an extra equation on the singular vector may be used as a quality control. In order to test this formulation the case of a rigid sphere has been considered at $ka = 6{\cdot}2832$. The condition number of the BEM coefficient matrix in this case was 12161, and a 30-node linear discretization has been used. The residual has been calculated as the vector containing the difference between the analytical magnitude of the nodal pressures and the magnitude of the nodal pressures calculated by the above described method. At $ka = 2\pi$ the interior nodal surface is a sphere with the same centre and the radius $a/2$. In Figure 3 is shown the error calculated as the length of the residual vector, and the value of the quality control calculated by equation (16b), as functions of the ρ co-ordinate of the CHIEF point, the z co-ordinate and the θ co-ordinate being zero; the sphere is centred in a cylindrical co-ordinate system (ρ, θ, z). It is evident that the quality control is very well correlated with the solution error.

7. SVD AT HIGHER FREQUENCIES

At higher frequencies the non-uniqueness problem becomes more severe due to the close spacing and to the "bandwidth" of the characteristic frequencies. One may very well encounter the situation in which the bands of bad solutions are no longer distinct and the solution is corrupted by two or more characteristic frequencies near any chosen frequency. This situation is reflected in two or more small singular values calculated by the SVD with corresponding singular vectors, and the (numerical) rank deficiency of the BEM coefficient matrix is greater than unity. The strategy now is for each singular vector to choose a CHIEF point that satisfies equation (16b) and thereby decreases the rank deficiency by one in the resulting rectangular matrix. When this has been done for all singular vectors the resulting rectangular matrix is fully ranked (the rank equals the column dimension) and hence the non-uniqueness problem is solved. Note that a CHIEF point has to "deal with" only *one* singular vector (the one it has been selected with respect to) and hence it is

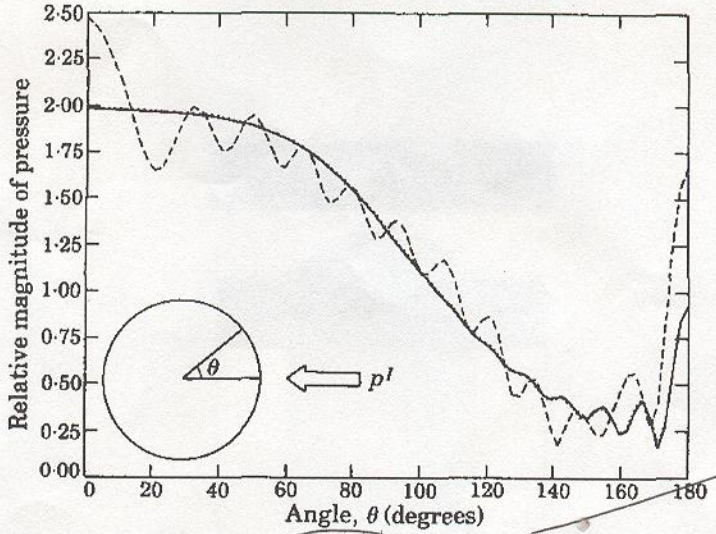


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Figure 4. Scattering by a rigid sphere for $ka = 15{\cdot}0397$, ——, Analytical solution; ----, BEM solution with one CHIEF point; ·····, BEM solution with two CHIEF points.

required only that equation (16b) is satisfied for this particular singular vector, whereas the CHIEF point does not need to satisfy equation (16b) for any other singular vector, because they are "dealt with" by other CHIEF points.

In order to provoke a higher rank deficiency, scattering of a plane wave with magnitude unity from a rigid sphere for $ka = 15{\cdot}0397$ has been considered. The generator of the sphere has been discretized in 79 elements and 80 nodes. In this case two singular values become very small due to the presence of another fictitious eigenfrequency at $ka = 15{\cdot}0335$ ($\kappa = w_1/w_N = 2466$; $w_1/w_{N-1} = 310$; $w_1/w_{N-2} = 9{\cdot}7$), and the numerical rank deficiency of the BEM coefficient matrix is two. The magnitude of the pressure on the surface is shown in Figure 4 as a function of the angle defined in the small inset in the figure. It is evident that in this case two "good" CHIEF points are required to obtain an accurate solution (the difference between the two curves is hardly noticeable). The CHIEF points are selected with respect to the quality control.

8. DISCUSSION

The two important features of the SVD are as follows: (1) the singular values, which allow the user to decide how many CHIEF points are needed (if any); (2) the singular vectors, which provide an excellent tool to check the quality of the CHIEF points.

The main disadvantages of the SVD are its great complexity and the fact that calculating the SVD is quite time consuming compared to other methods. However, in BEM the time consumed by setting up the equations is in most cases still larger than the time used to solve the system of equations.

Recently, another method to estimate the singular values and the singular vectors—the rank revealing QR factorization (RRQR)—has been discovered [17]. The RRQR is far more efficient (in terms of time consumed and storage) than the SVD.

In numerical linear algebra the work of solving a system of equations by using a given algorithm is often measured as the number of floating point operations (flops). A multiplication or an addition each involves one flop [18]. In order to solve an $n \times n$ system of equations by the means of LU decomposition [16] $2n^3/3$ flops are involved. The LU decomposition is the most efficient method of solving a general system of equations. The number of flops used by the SVD algorithm is $12n^3$ and the QR algorithm uses $4n^3/3$ flops.

However, when memory traffic is taken into account, the QR algorithm tends to be as effective as the normal LU decomposition algorithm. Moreover, the QR algorithm may readily be used to solve overdetermined systems in the least-squares sense. The rank revealing part of the RRQR factorization is a superstructure to the normal QR factorization involving only an order of n^2 flops. Hence the extra work of a RRQR factorization is negligible compared to that of a QR factorization. Once one has decided on the equations to be added to the original system of equations it is possible to update the QR factorization [18]. Hence it is not necessary to recalculate the factorization. In short, it can be stated that the tools described in this paper can be obtained without a significant amount of extra work, and they supply the user with an assurance of the quality of the solution obtained—and also near characteristic frequencies when CHIEF points are added.

Note that the attempt described in this paper is valid for any kind of extra equations that one may wish to add to the original BEM coefficient matrix. This attempt may therefore be used in more advanced formulations to circumvent the non-uniqueness problem, such as SuperCHIEF [13] or CHIEF-block [2]. The extra equations obtained by any of these advanced CHIEF methods may be checked in the same way as ordinary CHIEF points.

It must be emphasized that the test cases presented in this paper concern an axisymmetric model in which the rank deficiency problem is less severe than in a general three-dimensional formulation, but this method is valid for general three-dimensional formulations as well.

9. CONCLUSIONS

In this paper the non-uniqueness problem of the exterior BEM formulation has been investigated by means of singular value decomposition (SVD). It has been shown that the rank deficiency of the BEM coefficient matrix at characteristic frequencies may be revealed by the SVD.

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Furthermore, it has been shown that due to the "bandwidth" of the characteristic frequencies the rank deficiency of the BEM coefficient matrix may be greater than unity when two or several characteristic frequencies are near the frequency of interest.

The number of "good" CHIEF points needed to obtain a unique solution equals the rank deficiency of the BEM coefficient matrix, and it has been shown that by making use of the singular vectors obtained by the SVD the quality of the CHIEF points can be evaluated reliably.

This formulation may also be applied to more advanced methods to overcome the non-uniqueness problem.

The author believes that this formulation provides a useful tool not only for solving the non-uniqueness problem, but also for maintaining a mesh as coarse as possible for a given accuracy. This latter feature becomes very significant when modelling complex structures at higher frequencies.

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