Image solution for vertical motion of a point source towards a free surface

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The vertical constant-speed motion of a constant-strength point source towards a horizontal free boundary is analysed. A procedure based on expansions in even powers of the Froude number is employed. The asymptotic expansion of the potential is found to satisfy a simple differential equation, which, when integrated, yields an image-type solution valid for all Froude numbers. Froude-number effects are contained in a distribution of sources along the vertical line from the image of the submerged source with respect to the undisturbed free surface upward to infinity. The solution is valid for arbitrary values of the density ratio across the free surface.

1. Introduction

In the theoretical analysis of the hydrodynamics of bodies moving near a free water surface, it is convenient to work in terms of fundamental solutions of the governing Laplace equation and of the linearized free-surface boundary conditions. Solutions corresponding to the motion of a point source beneath the surface are of particular interest. By superposing these solutions such that the body surface is covered with sources, the problem is reduced to the determination of the strength of the body-bound source distribution so as to satisfy the body boundary condition.

The fundamental solution corresponding to the motion along an arbitrary path of a point source of time-dependent strength has been derived by Haskind (1946) and Brard (1948) and is discussed in Wehausen & Laitone's (1960) review article. The solution is derived by transform methods, and consists of three terms: the potential of the submerged source, of a singularity of equal but opposite strength at the image of the submerged source with respect to the undisturbed free surface, and of a superposition of standing waves of all wavelengths. The first two terms constitute the infinite-Froude-number approximation to the solution, while the effects of gravity on the fluid motion are contained in the third term.

The form of this last term, even in the special case of constant source strength and rectilinear source motion, is somewhat unwieldy. It would be preferable, at least from the interpretational (and possibly from the computational) point of view, if Froude-number effects were expressed as a superposition of sources or other singular solutions of the Laplace equation. Such an 'image' solution was found by Havelock (1927) for the two-dimensional problem in which a doublet of constant strength moves parallel to the water surface at constant speed. His image system consists of a horizontal distribution of doublets trailing rearward from the image point to infinity. The doublets are of constant strength, but their axes rotate harmonically along the length of the distribution.

In the present paper, an image solution is derived for the vertical constant-speed motion of a constant-strength point source towards the surface. The approach employed is somewhat unusual. The asymptotic expansion of the potential in even powers of the Froude number is derived, and is found to satisfy a simple first-order differential equation. The solution of this equation yields a final result, valid for all Froude numbers, in which Froude-number effects are contained in a vertical trail of sources from the image point upwards to infinity. The strength of the distribution decays exponentially with increasing distance from the image point. Both the maximum value and the rate of decay of the distribution strength are inversely proportional to the Froude number.

In the formulation of § 2 and the solution of § 3, the region above the surface is supposed to have zero density, as is approximately true in the case of an air-water interface. For completeness, the solution is extended in § 4 to the case where the fluid above the surface has an arbitrary density.

2. Formulation

The flow under consideration is assumed to be incompressible and irrotational. The problem may thus be formulated in terms of a velocity potential ϕ, defined so that its gradient yields the velocity field. From continuity, ϕ must satisfy the Laplace equation

$$\nabla^2\phi = 0 \tag{1}$$

everywhere in the flow field of interest, except at specified singular points. In particular, we seek a potential of the form

$$\phi = \frac{m}{r} + \phi_I, \tag{2}$$

where

$$r^2 = x^2 + (y-b)^2 + z^2. \tag{3}$$

The first term in (2) is the potential of a point source of strength $-4\pi m$.* The source is located at $(0, b, 0)$ in rectangular Cartesian coordinates (x, y, z), which are space-fixed, with the y-axis directed vertically upward. The plane $y = 0$ is the undisturbed position of a boundary which is free to distort itself under the influence of the source. The source is in vertical constant-speed motion beneath and towards the boundary, so that

$$\begin{aligned} b(t) &= b_0 + Ut \\ &< 0. \end{aligned} \tag{4}$$

Here U is the speed of the source and t is the time variable.

The second term in (2) is required to be harmonic in $y < \eta$, where $y = \eta(x, z, t)$ is the location of the free boundary. We assume the flow to be only slightly disturbed at this boundary. The dynamic condition of constant pressure on the free surface is then linearized and applied on the undisturbed position of the free surface, and takes the form

$$\phi_t(x, 0, z, t) + g\eta(x, z, t) = 0, \tag{5}$$

* The source strength, or volume rate of flow across any surface enclosing the source, is defined in this manner so as to make our terminology agree with that of Wehausen & Laitone (1960).

where the subscripts indicate partial differentiation and g is the acceleration due to gravity. Combining (5) with the kinematic relation between the surface motion and the fluid velocity at the surface, we obtain the linearized free-surface boundary condition on the potential,

$$\phi_{tt} + g\phi_y = 0 \quad \text{on} \quad y = 0. \tag{6}$$

The potential must also satisfy the requirement that the flow disturbances vanish far from the source

$$\phi_x, \phi_y, \phi_z \to 0 \quad \text{as} \quad r \to \infty. \tag{7}$$

Finally, we have the initial conditions,

$$\phi, \phi_t \to 0 \quad \text{on} \quad y = 0 \quad \text{as} \quad b \to -\infty. \tag{8}$$

It may be noted that equations (1), (2), and (6)–(8) determine a unique solution (Finkelstein 1957).

3. Image solution

It is convenient to rewrite the free-surface boundary condition (6) in the form

$$\phi_{bb} + \frac{1}{F}\phi_y = 0 \quad \text{on} \quad y = 0. \tag{9}$$

Here we have used (4) to replace t with b as the time variable, and have defined

$$F \equiv U^2/g. \tag{10}$$

Taking our characteristic length to be unity, we may refer to F as the square of the Froude number.

We seek an expansion of the potential in even powers of the Froude number $F^{\frac{1}{2}}$, and so assume a solution of the form

$$\phi_I = \sum_{n=0} F^n \phi_n. \tag{11}$$

We then substitute for ϕ in the governing equations (1), (7), (8) and (9) from equations (2) and (11) and equate terms of like order in F. Since m/r satisfies the Laplace equation (1) (except at the source point) and the boundary conditions (7) and (8) identically, so must ϕ_0, as well as all the other ϕ_n's.

On equating terms of order F^0 in the free-surface boundary condition (9), we obtain

$$\frac{\partial}{\partial y}\left[\phi_0 + \frac{m}{r}\right] = 0 \quad \text{on} \quad y = 0. \tag{12}$$

Thus, in the zeroth approximation, the problem is that of a point source near a plane wall. This has the well-known image solution

$$\phi_0 = \frac{m}{r_1}, \tag{13}$$

where

$$r_1^2 \equiv x^2 + (y+b)^2 + z^2. \tag{14}$$

Now collecting terms of order F^1 in equation (9), we may write the free-surface boundary condition on ϕ_1 as

$$\phi_{1_y} = -m\frac{\partial^2}{\partial b^2}\left[\frac{1}{r} + \frac{1}{r_1}\right] \quad \text{on} \quad y = 0, \tag{15}$$

or

$$\phi_{1_y} = -2\phi_{0_{yy}} \quad \text{on} \quad y = 0, \tag{16}$$

where the identity of equations (15) and (16) follows easily from equations (3), (15) and (14). Since ϕ_0 and ϕ_1 are without singularities in $y < 0$, the function $\phi_{1_y} + 2\phi_{0_{yy}}$ satisfies Laplace's equation everywhere in $y < 0$, vanishes on $y = 0$ and at infinity, and hence, from Green's theorem, vanishes everywhere in $y < 0$. Thus equation (16) may be continued into the region below the plane $y = 0$, and integrated to yield

$$\phi_1 = -2\phi_{0_y} = -2\phi_{0_b}, \tag{17}$$

where the equality of ϕ_{0_y} and ϕ_{0_b} is clear from equations (13) and (14).

Using similar arguments, we find by induction that

$$\phi_n = (-1)^n\, 2\frac{\partial^n}{\partial b^n}\phi_0 \quad \text{for} \quad n > 0. \tag{18}$$

Then, from (11) and (18),

$$\phi_I = \phi_0 + 2\sum_{n=1} (-F)^n \frac{\partial^n}{\partial b^n}\phi_0. \tag{19}$$

This expansion in even powers of the Froude number is only asymptotic (consider, for example, its value at $x = z = 0$). In general, asymptotic expansions may not be differentiated (Erdelyi 1956). However, by differentiating (9) with respect to b, regarding ϕ_{I_b} as the unknown, and repeating the above procedure, we find that the asymptotic expansion in F of ϕ_{I_b} is simply the b-derivative of (19), which we write as

$$\phi_{I_b} = -\phi_{0_b} - \frac{2}{F}\sum_{n=1} (-F)^n \frac{\partial^n}{\partial b^n}\phi_0. \tag{20}$$

Thus, from equations (19) and (20), we see that ϕ_I satisfies

$$\phi_{I_b} + \frac{1}{F}\phi_I = -\phi_{0_b} + \frac{1}{F}\phi_0. \tag{21}$$

The complementary solution of (21), of the form $e^{-b/F}\times$ any time-independent solution of Laplace's equation, is eliminated by application of equation (8). The particular solution is easily found, and, when substituted into (2), yields the result,

$$\phi = \frac{m}{r} - \frac{m}{r_1} + \frac{2m}{F}\int_{-b}^{\infty} e^{-(\xi+b)/F}\{x^2 + (y-\xi)^2 + z^2\}^{-\frac{1}{2}}\, d\xi, \tag{22}$$

which has the form advertised in § 1.

The solution for our special case is recovered from the general result reported by Wehausen & Laitone (1960) simply by integrating their equation (13.49) over time, with the result that

$$\phi = \frac{m}{r} - \frac{m}{r_1} + \frac{m}{\pi}\int_{-\pi}^{\pi} d\theta \int_0^{\infty} \frac{dk}{1+kF}\exp\{k[y+b] + ik[x\cos\theta + z\sin\theta]\}. \tag{23}$$

The equivalence of this result with our equation (22) is readily established by substituting the identity

$$\frac{1}{1+kF}e^{kb} = \frac{1}{F}e^{-b/F}\int_{-b}^{\infty} \exp\left\{-\left(k+\frac{1}{F}\right)\xi\right\} d\xi \tag{24}$$

into (23) and integrating over k and θ.

Thus we see that the known* solution (23) for vertical constant-speed motion of a constant-strength source could have been expressed in terms of images by introducing the trivial, though not obvious, substitution (24). Such a procedure was employed by Havelock (1927) in deriving the image solution for horizontal motion of a two-dimensional doublet beneath the surface. Nevertheless, the present procedure has the virtue of being more direct, and is felt to be of interest in itself.

Unfortunately, our procedure is not very powerful It has been possible to reproduce Havelock's (1927) solution, though not without difficulty. However, no success was obtained in attempts to express the solution for horizontal motion of a point singularity in terms of images, or to treat cases in which the source strength is variable.

4. Superposed-fluid problem

We now extend the solution of § 3 to the case in which the fluid above the free boundary $y = \eta$ has a finite density. The flow field of interest is then not restricted to the region $y < \eta$, but consists of all space.

We define ρ^+ and ρ^- as the fluid densities in the regions above and below the free surface, respectively. We anticipate discontinuities in the potential and in some of its derivatives across the free surface, and so define

$$\begin{aligned}\phi_I &= \phi_I^+ \quad \text{for} \quad y > \eta,\\ &= \phi_I^- \quad \text{for} \quad y < \eta.\end{aligned} \tag{25}$$

Both ϕ_I^+ and ϕ_I^- are functions defined everywhere, but which coincide with the potential ϕ_I only as required by equation (25). Where they do coincide with ϕ_I, ϕ_I^+ and ϕ_I^- are without singularities.

We neglect surface tension, so that the dynamic free-surface boundary condition is that the pressure be continuous across the surface. In its linearized form, this condition is [cf. equation (5)]

$$\rho^+\phi_t(x,0^+,z,t) - \rho^-\phi_t(x,0^-,z,t) + (\rho^+ - \rho^-)\,g\eta(x,z,t) = 0. \tag{26}$$

We also require that the velocity normal to the surface be continuous across it. Since m/r and all its derivatives are continuous across the surface, this condition may be written

$$\phi^+_{I_y} = \phi^-_{I_y} \quad \text{on} \quad y = 0. \tag{27}$$

Combining equations (26) and (27) with the kinematic relation between η_t and $\phi_y]_{y=0}$, we obtain a second free-surface boundary condition on the potential, which we write [cf. equation (9)]

$$\rho^+\phi^+_{I_{bb}} - \rho^-\phi^-_{I_{bb}} + \frac{\rho^+ - \rho^-}{F}\phi^-_{I_y} = m(\rho^- - \rho^+)\left[\frac{\partial^2}{\partial b^2} + \frac{1}{F}\frac{\partial}{\partial y}\right]\frac{1}{r} \quad \text{on} \quad y = 0. \tag{28}$$

* While the trivial specialization of the known solution for arbitrary motion of a variable-strength source to our case has not, to the author's knowledge, been published previously, Sakai, Husimi & Hatoyama (1933) derived a solution similar in form to (23) for the case of vertical motion in two dimensions of a line doublet.

The potential defined by equations (2), (3), and (25) is also required to satisfy (1), (7), and (8).

Equation (27) shows that ϕ_I may be represented by a vortex sheet on the undisturbed free surface. Then

$$\phi_I^+(x,0,z,b) = -\phi_I^-(x,0,z,b). \tag{29}$$

Differentiating (29) twice with respect to b, we may eliminate ϕ_I^+ from (28), which may then be written

$$\frac{\rho^-+\rho^+}{\rho^--\rho^+}\phi^-_{I_{bb}} + \frac{1}{F}\phi^-_{I_y} = -m\left[\frac{\partial^2}{\partial b^2} + \frac{1}{F}\frac{\partial}{\partial y}\right]\frac{1}{r}, \quad \text{on} \quad y = 0 \tag{30}$$

The solution for ϕ_I^- now proceeds as in §3. We determine the asymptotic expansions in F of ϕ_I^- and $\phi^-_{I_b}$ and show thereby that ϕ_I^- satisfies a simple first-order differential equation [cf. equation (21)], which, when integrated, yields the final result

$$\phi_I^- = -\frac{\rho^- - \rho^+}{\rho^- + \rho^+}\frac{m}{r_1} + \frac{2\rho^-}{\rho^- + \rho^+}\frac{m}{F'}\int_{-b}^{\infty} e^{-(\xi+b)/F'}\{x^2 + (y-\xi)^2 + z^2\}^{-\frac{1}{2}}\,d\xi, \tag{31}$$

where

$$F' \equiv \frac{\rho^- + \rho^+}{\rho^- - \rho^+}\frac{U^2}{g}. \tag{32}$$

From equation (29), the potential $\phi_I^+ + \phi_I^-$ vanishes on the plane $y = 0$. As is well known, a system of singularities satisfies such a condition if, for each source above the plane, a singularity of equal but opposite strength is positioned at its image with respect to the plane. Thus, taking note of equation (31), we may write down the solution for ϕ_I^+ without further calculation as

$$\phi_I^+ = \frac{\rho^- - \rho^+}{\rho^- + \rho^+}\frac{m}{r} - \frac{2\rho^-}{\rho^- + \rho^+}\frac{m}{F'}\int_{-\infty}^{b} e^{(\xi-b)/F'}\{x^2 + (y-\xi)^2 + z^2\}^{-\frac{1}{2}}\,d\xi. \tag{33}$$

This solution is readily extended to the case in which the source is moving in the low-density fluid downward towards the free boundary.

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