ON NON-UNIQUE SOLUTIONS OF WEAKLY SINGULAR INTEGRAL EQUATIONS IN PLANE ELASTICITY

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SUMMARY

A unique constant matrix is constructed for each smooth boundary curve, which generalizes the concept of logarithmic capacity and indicates when the Dirichlet problem of plane elasticity cannot be solved by means of integral equations of the first kind.

LET S be a domain in $\mathbb{R}^2$ bounded by a closed C^2-curve ∂S. The system of equations of plane elasticity can be written in the form

$$A(\partial_x)u(x) = q(x), \qquad x \in S, \tag{1}$$

where $A(\partial_x) = A(\partial/\partial x_1, \partial/\partial x_2)$ is the partial differential matrix operator defined by

$$A(\xi_1, \xi_2) = \begin{pmatrix} \mu\Delta + (\lambda+\mu)\xi_1^2 & (\lambda+\mu)\xi_1\xi_2 \\ (\lambda+\mu)\xi_1\xi_2 & \mu\Delta + (\lambda+\mu)\xi_2^2 \end{pmatrix}, \tag{2}$$

$u = (u_1, u_2)^T$ is the displacement vector, q is the body-force vector, $x = (x_1, x_2)$ is a generic point in $\mathbb{R}^2$ given in terms of its Cartesian coordinates, λ and μ are the Lamé constants of the (homogeneous and isotropic) material occupying the domain S, and $\Delta(\xi) = \xi_1^2 + \xi_2^2$. We also consider the boundary stress operator $T(\partial_x)$ defined by

$$T(\xi_1, \xi_2) = \begin{pmatrix} (\lambda+2\mu)\nu_1\xi_1 + \mu\nu_2\xi_2 & \mu\nu_2\xi_1 + \lambda\nu_1\xi_2 \\ \lambda\nu_2\xi_1 + \mu\nu_1\xi_2 & \mu\nu_1\xi_1 + (\lambda+2\mu)\nu_2\xi_2 \end{pmatrix},$$

where $\nu = (\nu_1, \nu_2)^T$ is the unit vector of the outward normal to ∂S.

In what follows we adopt some conventional notation:

(i) Greek and italic subscripts take the values 1, 2 and 1, 2, 3, respectively, and summation over repeated indices is understood.

(ii) We write $\mathcal{M}_{p\times q}$ for the space of $(p \times q)$-matrix functions; $H^{(i)}$ are the columns of a matrix $H \in \mathcal{M}_{p\times q}$, and E_n is the unit matrix in $\mathcal{M}_{n\times n}$.

(iii) If X is a space of scalar functions and $\varphi \in \mathcal{M}_{p\times q}$, then $\varphi \in X$ means that every component of φ belongs to X.

(iv) Let L be an operator defined on functions $\theta \in \mathcal{M}_{p\times 1}$ and such that $L\theta \in \mathcal{M}_{r\times 1}$. If $\Theta \in \mathcal{M}_{p\times q}$, then $L\Theta \in \mathcal{M}_{r\times q}$ is the matrix with $(L\Theta)^{(i)} = L\Theta^{(i)}$.

A well-known method of solution of the Dirichlet problem (u prescribed on ∂S) and of the Neumann problem (Tu prescribed on ∂S) for A in both the interior case, when $S = S^+$ is the finite domain enclosed by ∂S, and in the exterior case, when $S = S^- = \mathbb{R}^2 \backslash \bar{S}^+$, is based on the use of the single-layer and double-layer elastic potentials defined, respectively, by

$$(V\varphi)(x) = \int_{\partial S} D(x, y)\varphi(y)\, ds(y)$$

and

$$(W\varphi)(x) = \int_{\partial S} P(x, y)\varphi(y)\, ds(y).$$

Here

$$D(x, y) = A^*(\partial_x)t(x, y) \tag{3}$$

is a matrix of fundamental solutions for A constructed by means of Galerkin's representation, A^* is the adjoint of A,

$$t(x, y) = -\big(8\pi\mu(\lambda + 2\mu)\big)^{-1}|x - y|^2 \ln|x - y|, \tag{4}$$

and

$$P(x, y) = \big(T(\partial_y)D(y, x)\big)^T.$$

Since (by using a Newtonian potential, for example) we can reduce (1) to its homogeneous form, in what follows we assume that $q = 0$.

If $u \in C^2(S^+) \cap C^1(\bar{S}^+)$ is a solution of $Au = 0$, then the Somigliana formula (1) can be written in the form

$$V(Tu|_{\partial S}) - W(u|_{\partial S}) = \begin{cases} u & \text{in } S^+, \\ \tfrac{1}{2}u & \text{on } \partial S, \\ 0 & \text{in } S^-, \end{cases}$$

and shows that the solution of the Dirichlet problem is found throughout S^+ if $Tu|_{\partial S}$ can be computed. This procedure, known as the direct method, reduces therefore to solving uniquely the Fredholm integral equation of the first kind

$$V_0\varphi = f = \tfrac{1}{2}u|_{\partial S} + W_0(u|_{\partial S}), \tag{5}$$

where $V_0\theta$ and $W_0\theta$ are the direct values of $V\theta$ and $W\theta$ (the latter understood as principal value) on ∂S. For the two-dimensional Laplace equation there are smooth curves ∂S (of logarithmic capacity 1) on which the corresponding homogeneous equation (5) has non-zero solutions (see (**2** to **4**)). It is known (**5**) that this problem may also be encountered in plane elasticity. Below we show rigorously on which particular boundary contours this situation is certain to occur.

The next assertion gathers together a few properties of the elastic potentials (**1**, **6**) needed in the subsequent analysis.

THEOREM 1. (i) *If* $\varphi \in C(\partial S)$, *then* $V\varphi$ *and* $W\varphi$ *are analytic and satisfy* $A(V\varphi) = A(W\varphi) = 0$ *in* $S^+ \cup S^-$.

(ii) *If* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1)$, *then* $V_0\varphi$ *and* $W_0\varphi$ *exist, the functions*

$$\mathscr{V}^+(\varphi) = (V\varphi)|_{\bar{S}^+}, \qquad \mathscr{V}^-(\varphi) = (V\varphi)|_{\bar{S}^-}$$

are of class $C^{1,\alpha}(\bar{S}^+)$ *and* $C^{1,\alpha}(\bar{S}^-)$ *respectively, and*

$$T\mathscr{V}^+(\varphi) = (W_0^* + \tfrac{1}{2}I)\varphi, \qquad T\mathscr{V}^-(\varphi) = (W_0^* - \tfrac{1}{2}I)\varphi,$$

where W_0^* *is the adjoint of* W_0 *and* I *is the identity operator.*

(iii) *If* $\varphi \in C^{1,\alpha}(\partial S)$, $\alpha \in (0, 1)$, *then the functions*

$$\mathscr{W}^+(\varphi) = \begin{cases} (W\varphi)|_{S^+} & \text{in } S^+, \\ (W_0 - \tfrac{1}{2}I)\varphi & \text{on } \partial S, \end{cases} \qquad \mathscr{W}^-(\varphi) = \begin{cases} (W\varphi)|_{S^-} & \text{in } S^-, \\ (W_0 + \tfrac{1}{2}I)\varphi & \text{on } \partial S, \end{cases}$$

are of class $C^{1,\alpha}(\bar{S}^+)$ *and* $C^{1,\alpha}(\bar{S}^-)$ *respectively.*

(iv) $(W_0 + \tfrac{1}{2}I)\varphi = 0$ *if and only if* $\varphi = Fk$, *where the columns* $F^{(i)}$ *of the matrix*

$$F = \begin{pmatrix} 1 & 0 & x_2 \\ 0 & 1 & -x_1 \end{pmatrix}$$

form a basis for the space of rigid displacements and $k \in \mathscr{M}_{3\times 1}$ *is constant and arbitrary. (Clearly,* Fk *is an arbitrary rigid displacement.) Also,* $A(Fk) = 0$ *in* $\mathbb{R}^2$ *and* $T(Fk) = 0$ *on* ∂S.

(v) *The solutions of the equation* $(W_0^* + \tfrac{1}{2}I)\varphi = 0$ *form a subspace of* $C^{1,\alpha}(\partial S)$ *of dimension* 3. *(For convenience, we write* $\varphi = Gl$, *where the columns* $G^{(i)}$ *of* $G \in \mathscr{M}_{2\times 3}$ *are linearly independent and* $l \in \mathscr{M}_{3\times 1}$ *is constant and arbitrary.)*

(vi) $\tfrac{1}{4}$ *is an eigenvalue of* W_0^{*2}, *and the corresponding eigenspace coincides with that of* W_0^* *for the eigenvalue* $-\tfrac{1}{2}$ *(see* (v) *above).*

(vii) *Let* N_0 *be the operator defined on* $C^{1,\alpha}(\partial S)$ *by* $N_0\varphi = T\mathscr{W}^+(\varphi)$. *If* $N_0\varphi = 0$, *then* $\varphi = Fk$, *where* $k \in \mathscr{M}_{3\times 1}$ *is constant and arbitrary.*

(viii) $N_0V_0 = W_0^{*2} - \tfrac{1}{4}I$ *on* $C^{0,\alpha}(\partial S)$.

(ix) *Let* $\mathscr{A}$ *be the class of functions* $u \in \mathscr{M}_{2\times 1}$ *which, as* $r = |x| \to \infty$, *admit an asymptotic expansion of the form*

$$u_1(r, \theta) = r^{-1}(\alpha m_0 \sin\theta + m_1 \cos\theta + m_0 \sin 3\theta + m_2 \cos 3\theta) + O(r^{-2}),$$

$$u_2(r, \theta) = r^{-1}(m_3 \sin\theta + \alpha m_0 \cos\theta + m_4 \sin 3\theta - m_0 \cos 3\theta) + O(r^{-2}),$$

where $m_0, \dots, m_4$ *are arbitrary constants and* $\alpha = (\lambda + 3\mu)/(\lambda + \mu)$. *Also, let* $\mathscr{A}^*$ *be the class of functions of the form* $u = Fk + \sigma^{\mathscr{A}}$, *with* $\sigma^{\mathscr{A}} \in \mathscr{A}$. *Then* $W\varphi \in \mathscr{A}$ *and*

$$V\varphi = M^\infty(p\varphi) + \sigma^{\mathscr{A}},$$

where p is the operator defined on continuous functions $\varphi \in \mathcal{M}_{2\times 1}$ on ∂S by

$$p\varphi = \int_{\partial S} F^T \varphi \, ds$$

and

$$4\pi\mu(\alpha+1)M^{\infty}(r,\theta) = \begin{pmatrix} -2\alpha(\ln r + 1) + \cos 2\theta & \sin 2\theta & r^{-1}(\alpha+1)\sin\theta \\ \sin 2\theta & -2\alpha(\ln r + 1) - \cos 2\theta & -r^{-1}(\alpha+1)\cos\theta \end{pmatrix},$$

with $AM^{\infty} = 0$ in $\mathbb{R}^2$.

(x) *The interior Dirichlet problem has at most one solution $u \in C^2(S^+) \cap C^1(\bar{S}^+)$.*

(xi) *The exterior Dirichlet problem has at most one solution $u \in C^2(S^-) \cap C^1(\bar{S}^-) \cap \mathcal{A}^*$. If $u|_{\partial S} \in C^{1,\alpha}(\partial S)$ and G can be chosen so that the sets $\{F^{(i)}\}$ and $\{G^{(i)}\}$ are biorthonormal, that is, $\int_{\partial S} F^T G \, ds = E_3$, then this problem does, in fact, have a unique such solution, which can be expressed as the sum of a double-layer potential and a specific rigid displacement Fk, with $k = \int_{\partial S} G^T u|_{\partial S} \, ds$.*

The concept of logarithmic capacity is discussed thoroughly in (7), where it is shown that this number occurs naturally in problems involving the Laplace operator and the single-layer potential. Analogues of such statements can also be proved in plane elasticity. In this case, however, Robin's constant associated with logarithmic capacity is replaced by a constant (3×3)-matrix, and the construction techniques have to be modified, since differentiation along ∂S and a Somigliana formula for functions with logarithmic growth at infinity, essential to the considerations in (7), are inappropriate in two-dimensional elasticity.

We introduce this generalization in two different ways.

THEOREM 2. *For any closed C^2-curve ∂S and any $\alpha \in (0, 1)$, there are a unique $\Phi \in \mathcal{M}_{2\times 3} \cap C^{1,\alpha}(\partial S)$ and a unique constant $\mathscr{C} \in \mathcal{M}_{3\times 3}$ such that the $\Phi^{(i)}$ are linearly independent and*

$$V_0\Phi = F\mathscr{C}, \qquad p\Phi = E_3.$$

Proof. By Theorem 1(v), (vi), (viii), we can write

$$(W_0^{*2} - \tfrac{1}{4}I)G = 0 = N_0(V_0 G),$$

and from Theorem 1(vii) we deduce that $V_0 G = FK$ for some constant $K \in \mathcal{M}_{3\times 3}$.

Let $H = pG = \int_{\partial S} F^T G \, ds$, and suppose that $\det H = 0$. Then there is a constant non-zero $h \in \mathcal{M}_{3\times 1}$ such that $Hh = 0$. In view of Theorem 1(i), (v), (ix), the function $U = V(Gh) - FKh$ satisfies

$$AU = 0 \quad \text{in } S^+ \cup S^-,$$
$$U = (V_0G - FK)h = 0 \quad \text{on } \partial S,$$
$$U = M^\infty(pG)h + \sigma^{\mathscr{A}} - FKh = M^\infty(Hh) + \sigma^{\mathscr{A}} - FKh$$
$$= -FKh + \sigma^{\mathscr{A}} \quad \text{as } |x| \to \infty.$$

By Theorem 1(x), (xi), $U = V(Gh) - FKh = 0$ in $S^+ \cup S^-$. Since FKh is a rigid displacement and $U = 0$ on ∂S, it follows that $FKh = 0$, hence, $V(Gh) = 0$ in $S^+ \cup S^-$. Theorem 1(ii) now yields $Gh = 0$, which contradicts the linear independence of the $G^{(i)}$. Consequently, $\det H \neq 0$, and, since H is constant, we see that

$$V_0(GH^{-1}) = (V_0G)H^{-1} = FKH^{-1},$$
$$p(GH^{-1}) = (pG)H^{-1} = HH^{-1} = E_3,$$

so we can take $\Phi = GH^{-1}$ and $\mathscr{C} = KH^{-1}$ as a solution pair for our problem.

To prove uniqueness, let Φ_1, $\mathscr{C}_1$ and Φ_2, $\mathscr{C}_2$ be two such solutions. Writing $\Phi = \Phi_1 - \Phi_2$ and $\mathscr{C} = \mathscr{C}_1 - \mathscr{C}_2$, and taking into account the fact that $p\Phi = 0$, we use Theorem 1(i), (iv), (ix) again to see that

$$A(V\Phi - F\mathscr{C}) = 0 \quad \text{in } S^+ \cup S^-,$$
$$V_0\Phi - F\mathscr{C} = 0 \quad (\text{on } \partial S),$$
$$V\Phi - F\mathscr{C} = \Sigma^{\mathscr{A}} \quad \text{as } |x| \to \infty.$$

As above, this yields $V\Phi - F\mathscr{C} = 0$ in $S^+ \cup S^-$, which in turn leads to $F\mathscr{C} = 0$. Since the $F^{(i)}$ are linearly independent, we conclude that $\mathscr{C} = 0$ and, by means of the usual argument involving Theorem 1(ii), also that $\Phi = 0$.

REMARK 1. The $\Phi^{(i)} = (GH^{-1})^{(i)}$ form a basis for the eigenspace of W_0^* corresponding to the eigenvalue $-\frac{1}{2}$. Theorem 2 shows that the sets $\{F^{(i)}\}$ and $\{\Phi^{(i)}\}$ are biorthonormal.

REMARK 2. If $\det \mathscr{C} \neq 0$ and $V_0\varphi = Fc$ with a constant $c \in \mathscr{M}_{3\times 1}$, then $c = \mathscr{C}h$ and $\varphi = \Phi h$ for some constant $h \in \mathscr{M}_{3\times 1}$. For $\det \mathscr{C} \neq 0$ implies that $\{\mathscr{C}^{(i)}\}$ is a basis for the space of constant elements in $\mathscr{M}_{3\times 1}$, so $c = \mathscr{C}h$ for some constant $h \in \mathscr{M}_{3\times 1}$. Then, by Theorem 2, $V_0\varphi = (F\mathscr{C})h$, therefore, $V_0(\varphi - \Phi h) = 0$. Since $\det \mathscr{C} \neq 0$, we conclude that $\varphi = \Phi h$ (6).

THEOREM 3. *For every closed C^2-curve ∂S, there are a unique function $\Psi \in \mathscr{M}_{2\times 3}$ and a unique constant $\bar{\mathscr{C}} \in \mathscr{M}_{3\times 3}$ such that*

$$A\Psi = 0 \quad \textit{in } S^-,$$
$$\Psi = 0 \quad \textit{on } \partial S,$$
$$\Psi = M^\infty - F\bar{\mathscr{C}} + \Sigma^{\mathscr{A}} \quad \textit{as } |x| \to \infty.$$

Proof. We set $\bar{\Psi} = \Psi - M^\infty$. Since $AM^\infty = 0$ in S^- (see Theorem 1(ix)), the given problem becomes

$$A\bar{\Psi} = 0 \quad \text{in } S^-,$$

$$\bar{\Psi} = -M^\infty \quad \text{on } \partial S,$$

$$\bar{\Psi} = -F\bar{\mathscr{C}} + \Sigma^{\mathscr{A}} \quad \text{as } |x| \to \infty.$$

This is an exterior Dirichlet problem (for each $\bar{\Psi}^{(i)}$), which, according to Theorem 1(xi) and Remark 1, has a unique solution

$$\bar{\Psi} = \mathscr{W}^-(\tilde{\Phi}) - F\bar{\mathscr{C}},$$

where $\tilde{\Phi} \in \mathscr{M}_{2\times 3} \cap \mathscr{C}^{1,\alpha}(\partial S)$ and $\bar{\mathscr{C}} \in \mathscr{M}_{3\times 3}$ is constant and uniquely defined by

$$\bar{\mathscr{C}} = \int_{\partial S} \Phi^T M^\infty \, ds,$$

with Φ given by Theorem 2. Consequently, by Theorem 1(ix), the function

$$\Psi = \bar{\Psi} + M^\infty = M^\infty - F\bar{\mathscr{C}} + \mathscr{W}^-(\tilde{\Phi}) = M^\infty - F\bar{\mathscr{C}} + \Sigma^{\mathscr{A}}$$

completes a solution pair for the given problem.

The uniqueness of this solution pair is shown in the usual way. Let Ψ_1, $\bar{\mathscr{C}}_1$ and $\Psi_2, \bar{\mathscr{C}}_2$ be two such solutions. Then the pair $\Psi = \Psi_1 - \Psi_2$, $\bar{\mathscr{C}} = \bar{\mathscr{C}}_1 - \bar{\mathscr{C}}_2$ satisfies

$$A\Psi = 0 \quad \text{in } S^-,$$

$$\Psi = 0 \quad \text{on } \partial S,$$

$$\Psi = -F\bar{\mathscr{C}} + \Sigma^{\mathscr{A}} \quad \text{as } |x| \to \infty,$$

and, by Theorem 1(xi), $\Psi = 0$ in S^-. This implies that $F\bar{\mathscr{C}} = 0$, which, in turn, since the $F^{(i)}$ are linearly independent, leads to $\bar{\mathscr{C}} = 0$.

THEOREM 4. *The pairs $\Phi, \mathscr{C}$ and $\Psi, \bar{\mathscr{C}}$ in Theorems 2 and 3 are connected by the relations*

$$\mathscr{C} = \bar{\mathscr{C}}, \qquad \Psi = V\Phi - F\mathscr{C}, \qquad \Phi = -T\Psi. \tag{6}$$

Proof. By Theorems 1(i), (ix) and 2, $\tilde{\Psi} = V\Phi - F\mathscr{C}$ satisfies

$$A\tilde{\Psi} = 0 \quad \text{in } S^-,$$

$$\tilde{\Psi} = 0 \quad \text{on } \partial S,$$

$$\tilde{\Psi} = M^\infty(p\Phi) + \Sigma^{\mathscr{A}} - F\mathscr{C} = M^\infty - F\mathscr{C} + \Sigma^{\mathscr{A}} \quad \text{as } |x| \to \infty,$$

which is the problem in Theorem 3. Since the latter has a unique solution, it follows that

$$\mathscr{C} = \tilde{\mathscr{C}}, \qquad \Psi = \tilde{\Psi} = V\Phi - F\mathscr{C}.$$

According to Remark 1, $(W_0^* + \frac{1}{2}I)\Phi = 0$, or $W_0^*\Phi = -\frac{1}{2}\Phi$. Hence, from $(6)_2$ and Theorem 1(ii) we find that

$$T\Psi = T\mathscr{V}^-(\Phi) = (W_0^* - \tfrac{1}{2}I)\Phi = -\tfrac{1}{2}\Phi - \tfrac{1}{2}\Phi = -\Phi,$$

as required.

THEOREM 5. *The equation* $V_0\varphi = 0$ *has non-zero solutions if and only if* ∂S *is such that* $\det \mathscr{C} = 0$.

Proof. If $\det \mathscr{C} = 0$, then $\mathscr{C}h = 0$ for some constant non-zero $h \in \mathscr{M}_{3\times 1}$. By Theorem 2, there is $\Phi \in \mathscr{M}_{2\times 3}$ such that the $\Phi^{(i)}$ are linearly independent and $V_0\Phi = F\mathscr{C}$. Therefore,

$$V_0(\Phi h) = (V_0\Phi)h = (F\mathscr{C})h = F(\mathscr{C}h) = 0,$$

with $\Phi h \neq 0$, since the $\Phi^{(i)}$ are linearly independent.

If $\det \mathscr{C} \neq 0$, then, by Theorem 2, $V\varphi + (F\mathscr{C} - V\Phi)p\varphi$ is a solution in $\mathscr{A}^*$ of the homogeneous exterior Dirichlet problem since, as $|x| \to \infty$,

$$V\varphi + (F\mathscr{C} - V\Phi)p\varphi = F\mathscr{C}p\varphi + \sigma^{\mathscr{A}}.$$

Hence, by Theorem 1(xi), $V\varphi + (F\mathscr{C} - V\Phi)p\varphi = 0$ in S^-, which means that $F\mathscr{C}p\varphi = 0$. In view of the linear independence of the $F^{(i)}$ and the assumption that $\det \mathscr{C} \neq 0$, this yields $p\varphi = 0$. Consequently, by Theorem 1(ix), $V\varphi \in \mathscr{A}$. We now use the fact that $V\varphi$ is a solution of both the interior and exterior Dirichlet problems to conclude that $V\varphi = 0$ in $\mathbb{R}^2$, which, in turn, leads to $\varphi = 0$. Thus, if the equation $V_0\varphi = 0$ has non-zero solutions, then we must necessarily have $\det \mathscr{C} = 0$.

EXAMPLE. From (3), (2) and (4) we find that

$$D_{\alpha\beta} = -\frac{(\lambda + \mu)}{8\pi\mu(\lambda + 2\mu)}\left[(2\alpha \ln|x - y| + 2\alpha + 1)\delta_{\alpha\beta} - 2\frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)}{|x - y|^2}\right],$$

where $\delta_{\alpha\beta}$ is Kronecker's symbol. Let ∂S be the circle with the centre at the origin and radius R. Since for this choice

$$\int_{\partial S} \ln|x - y|\, ds(y) = 2\pi R \ln R, \qquad \int_{\partial S} \frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)}{|x - y|^2}\, ds(y) = \pi R\delta_{\alpha\beta},$$

we see that

$$\int_{\partial S} D(x, y)\, ds(y) = -\frac{\lambda + 3\mu}{\mu(\lambda + 2\mu)} R(\ln R + 1)E_2.$$

This implies that if $R = e^{-1}$, then every constant $\varphi \in \mathcal{M}_{2\times 1}$ satisfies $V_0\varphi = 0$. In fact, the full calculation yields

$$\Phi = \frac{e}{2\pi}\begin{pmatrix} 1 & 0 & \sin\theta \\ 0 & 1 & -\cos\theta \end{pmatrix}, \qquad \mathscr{C} = \frac{e^2}{4\pi\mu}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As expected, $\mathscr{C}$ is singular. These results coincide with those in (**8**).

The question of non-zero solutions of the equation $V_0\varphi = 0$ was also mentioned in (**9**), where their existence for certain boundary curves seems to have been overlooked. The matrix of fundamental solutions used there is somewhat different, namely

$$D_{\alpha\beta}(x, y) = \delta_{\alpha\beta} \ln|x - y| - \frac{1}{\alpha}\frac{(x_\alpha - y_\alpha)(x_\beta - y_\beta)}{|x - y|^2}.$$

For the same choice of ∂S as above, this yields

$$\int_{\partial S} D(x, y)\, ds(y) = \pi R\left(2 \ln R - \frac{1}{\alpha}\right)E_2,$$

which means that if

$$R = e^{1/(2\alpha)} = e^{(\lambda + \mu)/(2(\lambda + 3\mu))},$$

then every constant $\varphi \in \mathcal{M}_{2\times 1}$ is a solution of $V_0\varphi = 0$.

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