A relation between the logarithmic capacity and the condition number of the BEM-matrices

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Abstract

We establish a relation between the logarithmic capacity of a two-dimensional domain and the solvability of the boundary integral equation for the Laplace problem on that domain. It is proved that when the called logarithmic capacity is equal to one the boundary integral equation does not have a unique solution. A similar result is derived for the linear algebraic systems that appear in the boundary element method. As these systems are based on the boundary integral equation, no unique solution exists when the logarithmic capacity is equal to one. Hence the system matrix is ill-conditioned. We give several examples to illustrate this and investigate the analogies between the Laplace problem with Dirichlet and mixed boundary conditions.

1 Introduction

Boundary integral equations (BIE) form the basis for the boundary element method (BEM), which turns the BIE into a linear system of algebraic equations. The succes of the solution process of the linear system depends to a large extent on the condition number of the corresponding system matrix. Therefore it is important to have an a priori estimate for the magnitude of the condition number. To retrieve information about this condition number we have to resort to information resulting from the boundary integral formulation. If the BIE does not have a unique solution also the system of equations in the BEM does not have a unique solution, and the corresponding system matrix is ill-conditioned.

For the uniqueness of the solution of the BIE arising from a Laplace equation some interesting results can be found in literature. In [9], [7] and [15] it is observed that the BIE for the Laplace equation with Dirichlet boundary conditions does not have a unique solution if the scaling of the domain is inappropriate. This introduces an extraordinary phenomenon: the scaling of a domain affects the uniqueness properties of the solution of the BIE. Consider the BIE on a unit square domain, for instance, and rescale the domain to an arbitrary size. For almost all scalings the problem will have a unique solution, but there is one particular scaling for which this is not true. An intriguing question is whether we can know this scaling beforehand.

The remedy is to choose a scaling such that a unique solution does exist. It turns out that this is achieved when the Euclidean diameter of the domain is smaller than one. The authors in [18] give an explanation for this, using the concept of logarithmic capacity. They prove that if the logarithmic capacity of a domain is equal to one, then the boundary integral operator is not positive definite, and consequently no unique solution exists. It is shown that this logarithmic capacity is strongly related to the Euclidean diameter, see [10] and [12]. Unfortunately, for very few domains the logarithmic capacity can be calculated explicitly. However, upper and lower bounds exist [1], [11] and also numerically computed estimates can be found [13].

As of yet, the Laplace problem with mixed boundary conditions received little attention . In [8] and [14] it is stated that this problem may not be uniquely solvable if the logarithmic capacity is equal to one, but this statement is not clarified any further. Therefore the topic of this paper is existence of a unique solution of the BIE for the Laplace equation with mixed boundary conditions in relation with the logarithmic capacity. We are aware of several formulations of the boundary integral equations. In this paper we choose for the direct symmetric collocation formulation. The direct formulation involves functions that can be easily related to physical quantities, whereas the indirect formulation uses auxiliary functions that have no physical meaning. The symmetric formulation, involving the single and double layer potentials, is more commonly used than the asymmetric formulation, which incorporates the hypersingular operator. Moreover, the asymmetric formulation yields matrices whose condition numbers are insensible to rescaling the domain. We prefer the collocation method above the Galerkin method. Again the collocation method is more commonly used and it does not require a second integration step like Galerkin method does.

In Section 2 we briefly outline the various problems that are the topic of interest in this paper. The logarithmic capacity is introduced in Section 3. Section 4 lists the main results with respect to the uniqueness properties of these problems. In Section 5 we illustrate the findings from the fourth section.

2 Setting

In this section we present a brief survey of the various problems we are studying in this paper. Let Ω be a simply connected domain in $2D$ whose boundary Γ is a closed curve. In the interior of Ω the Laplace equation holds for the unknown function $u = u(\boldsymbol{x})$,

$$\nabla^2 u = 0,\ \boldsymbol{x} \in \Omega. \tag{1}$$

The fundamental solution G of the Laplace operator ∇^2 is given by

$$G(\boldsymbol{x}, \boldsymbol{y}) := \frac{1}{2\pi} \log \frac{1}{\|\boldsymbol{x} - \boldsymbol{y}\|}. \tag{2}$$

We denote by q the derivative of u with respect to the outward normal $\boldsymbol{n}$ at Γ. Introduce the single and double layer potential by

$$\begin{aligned} (\mathcal{K}^s q)(\boldsymbol{x}) &:= \int_\Gamma G(\boldsymbol{x}, \boldsymbol{y}) q(\boldsymbol{y}) d\Gamma_y,\ \boldsymbol{x} \in \Gamma, \\ (\mathcal{K}^d u)(\boldsymbol{x}) &:= \int_\Gamma \frac{\partial}{\partial n_y} \{G(\boldsymbol{x}, \boldsymbol{y})\}\, u(\boldsymbol{y}) d\Gamma_y,\ \boldsymbol{x} \in \Gamma, \end{aligned} \tag{3}$$

respectively, and let $\mathcal{I}$ be the identity operator. Then the boundary integral equation for the Laplace equation reads (cf. [2])

$$\frac{1}{2} u + \mathcal{K}^d u = \mathcal{K}^s q, \boldsymbol{x} \in \Gamma. \tag{4}$$

At each point at the boundary we either prescribe u or q. We distinguish three different problems.
Dirichlet problem

$$u = \tilde{u},\ \boldsymbol{x} \in \Gamma. \tag{5}$$

Mixed problem

$$\begin{aligned} u &= \tilde{u},\ \boldsymbol{x} \in \Gamma_1, \\ q &= \tilde{q},\ \boldsymbol{x} \in \Gamma_2, \end{aligned} \tag{6}$$

Neuman problem

$$q = \tilde{q},\ \boldsymbol{x} \in \Gamma. \tag{7}$$

where $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.
For each of these three problems we will investigate the existence of a unique solution.

3 Logarithmic capacity

To study the uniqueness properties of the Dirichlet and the mixed problem in the next section we need to introduce the notion of logarithmic capacity. A more detailed study can be found in [6] and [16]. Let σ be a Borel measure on a set E. We define the energy integral I by

$$I(E) := \int_E \int_E \log \frac{1}{\|\boldsymbol{x}-\boldsymbol{y}\|} d\sigma(\boldsymbol{x}) d\sigma(\boldsymbol{y}). \tag{8}$$

The logarithmic capacity $C_l(E)$ is related to the infimum over all Borel measures of this integral by

$$-\log C_l(E) := \inf_\sigma I(\sigma). \tag{9}$$

In this paper functions are defined on the boundary Γ of a domain Ω. So instead of using general sets E we work with the logarithmic capacity related to the the boundary curve Γ. In [18] it is noted that if Γ is suitably regular $d\sigma(\boldsymbol{x})$ may be replaced by $q(\boldsymbol{x})d\Gamma_x$. In this paper we assume that the boundary Γ satisfies this demand. Then we may redefine the energy integral as

$$I(q) := \int_\Gamma \int_\Gamma \log \frac{1}{\|\boldsymbol{x}-\boldsymbol{y}\|} q(\boldsymbol{x}) q(\boldsymbol{y}) d\Gamma_x d\Gamma_y \tag{10}$$

and the logarithmic capacity is related to this integral by

$$-\log C_l(\Gamma) := \inf_q I(q). \tag{11}$$

Here the infimum is taken over all functions q in $L_1(\Gamma)$ with the restriction that

$$\int_\Gamma q(\boldsymbol{x}) d\Gamma_x = 1. \tag{12}$$

Let us give a physical interpretation of the logarithmic capacity. For simplicity let the domain Ω be contained in the disc with radius $1/2$. In that case it can be shown that the integral $I(q)$ is positive. The function q can be seen as a charge distribution over a conducting domain Ω. Faraday demonstrated that this charge will only reside at the exterior boundary of the domain, in our case at Γ. We normalize q in such a way that the total amount of charge at Γ is equal to one cf. condition (12). The function $\mathcal{K}^s q$ is the potential due to the charge distribution q. Note that the integral I can also be written as

$$I(q) = 2\pi \int_\Gamma (\mathcal{K}^s q)(\boldsymbol{x}) q(\boldsymbol{x}) d\Gamma_x. \tag{13}$$

Hence I can be seen as the energy of the charge distribution q. The charge will distribute itself over Γ in such a way that the energy I is minimized. So the quantity $-\log C_l(\Gamma)$ is the minimal amount of energy. Hence the logarithmic capacity $C_l(\Gamma)$ is a measure for the capability of the geometry Γ to support a certain amount of charge.

For most curves the logarithmic capacity is not known explicitly. Only for a few elementary shapes the logarithmic capacity can be calculated; we have listed some in Table 1.

There are also some useful properties [1, 6] that help us to determine or estimate the logarithmic capacity.

1. If Γ is the outer boundary of a closed bounded domain Ω, then $C_l(\Gamma) = C_l(\Omega)$. This agrees with the idea of Faraday's cage, mentioned above.

2. Denote by d_Γ the Euclidean diameter of Ω, then $C_l(\Gamma) \leq d_\Gamma$. Hence the radius of the smallest circle in which Γ is contained is an upper bound for the logarithmic capacity of Γ.

3. If $\Gamma = \boldsymbol{x} + \alpha\Gamma_1$, then $C_l(\Gamma) = \alpha C_l(\Gamma_1)$. Hence the logarithmic capacity behaves linearly with respect to scaling and is invariant with respect to translation.

4. If $\Omega_1 \subset \Omega_2$, then $C_l(\Omega_1) \leq C_l(\Omega_2)$.

5. For a convex domain Ω,

$$C_l(\Omega) \geq \left(\frac{\text{area}(\Omega)}{\pi}\right)^{1/2}. \tag{14}$$

Table 1: The logarithmic capacity of some sets.

set Γ	logarithmic capacity $C_l(\Gamma)$
circle with radius R	R
square with side L	$\frac{\Gamma(\frac{1}{4})^2}{4\pi^{3/2}} L \approx 0.59017 \cdot L$
ellipse with semi-axes a and b	$(a+b)/2$
interval of length a	$\frac{1}{4}a$
isosceles right triangle side a	$\frac{3^{3/4}\Gamma^2(1/4)}{2^{7/2}\pi^{3/2}}\, a \approx 0.476\, a$

4 Uniqueness results

For each of the three problems described in Section 2 we elaborate on the existence of a unique solution.

4.1 Dirichlet problem

For the BIE that arises from the Laplace equation with Dirichlet boundary conditions we have the following result.

Theorem 1 *There exists a nonzero q_e in $L_1(\Gamma)$ such that*

$$(\mathcal{K}^s q_e)(\boldsymbol{x}) = -\frac{1}{2\pi}\log C_l(\Gamma),\ \boldsymbol{x} \in \Gamma. \tag{15}$$

Proof. The proof that for any curve Γ there is a nonzero function q_e such that $\mathcal{K}^s q_e = 0$ can be found in [18, 6]. ■

In the following we briefly mention the major steps in the proof. We observe that for the values of the energy integral (8) we have $-\infty < I(q) \leq \infty$. If the infimum of the energy integral is infinitely large, then by definition the logarithmic capacity is equal to zero.
Suppose that $C_l(\Gamma) > 0$ and thus $-\infty \leq I(q) \leq \infty$. We define the function space $\bar{L}_1(\Gamma)$ by

$$\bar{L}_1(\Gamma) := \left\{ q \in L_1(\Gamma) | \int_\Gamma q(\boldsymbol{x}) d\Gamma = 1 \right\}. \tag{16}$$

In [6, p. 282] it is proven that for each curve Γ there exists a unique q_e such that

$$I(q_e) = \inf_{q \in \bar{L}_1(\Gamma)} I(q) = -\log C_l(\Gamma). \tag{17}$$

In terms of Borel measures this minimizing function q_e is often called the *equilibrium distribution*. The result in (17) guarantees that for each curve Γ an equilibrium distribution exists.
Let Γ be a closed bounded domain with positive logarithmic capacity and a connected complement. Again q_e denotes the equilibrium distribution. In [6, p. 287] it is proven that $2\pi\mathcal{K}^s q_e \leq -\log C_l(\Gamma)$ in the whole plane and $2\pi\mathcal{K}^s q_e = -\log C_l(\Gamma)$ at Γ, except possibly for a subset which has zero logarithmic capacity.

Theorem 1 leads to the following result.

Corollary 1 *If $C_l(\Gamma) = 1$ there exists a nonzero q_e such that $\mathcal{K}^s q_e = 0$.*

Thus in the specific case that $C_l(\Gamma) = 1$ the single layer operator $\mathcal{K}^s$ admits an eigenfunction q_e with zero eigenvalue. Hence $\mathcal{K}^s$ is not positive definite and the Dirichlet problem does not have a unique solution.

If we rescale the domain such that the Euclidean diameter is smaller than one, then the second property in Section 3 shows us that the logarithmic capacity will also be smaller than one. In this way we can guarantee the existence of a unique solution of the BIE. However it may not be desirable to perform such a scaling.

Another possibility to obtain uniqueness is to supplement the BIE with the condition

$$\int_\Gamma q(\boldsymbol{x})d\Gamma = 0, \tag{18}$$

which follows from the fact that u is a harmonic function in Ω. Since the contour integral of q_e is equal to one, q_e can never be a solution of the BIE supplemented with (18).

4.2 Mixed problem

For the Laplace problem with mixed boundary conditions we have to rewrite the BIE in (4). For $i = 1, 2$ we introduce the functions $u_i := u|_{\Gamma_i}$ and $q_i := q|_{\Gamma_i}$ and the boundary integral operators

$$(\mathcal{K}_i^s q)(\boldsymbol{x}) \quad := \quad \int_{\Gamma_i} G(\boldsymbol{x},\boldsymbol{y})q(\boldsymbol{y})d\Gamma_y,\ \boldsymbol{x}\in\Gamma, \tag{19a}$$

$$(\mathcal{K}_i^d u)(\boldsymbol{x}) \quad := \quad \int_{\Gamma_i} \frac{\partial}{\partial n_y}G(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})d\Gamma_y,\ \boldsymbol{x}\in\Gamma. \tag{19b}$$

Note that with (6) $u_1 = \tilde{u}$ and $q_2 = \tilde{q}$. Now we write (4) as a system of two BIE's,

$$\mathcal{K}_2^d u_2 - \mathcal{K}_1^s q_1 \quad = \quad \mathcal{K}_2^s\tilde{q} - \frac{1}{2}\tilde{u} - \mathcal{K}_1^d\tilde{u},\ \boldsymbol{x}\in\Gamma_1, \tag{20a}$$

$$\frac{1}{2}u_2 + \mathcal{K}_2^d u_2 - \mathcal{K}_1^s q_1 \quad = \quad \mathcal{K}_2^s\tilde{q} - \mathcal{K}_1^d\tilde{u},\ \boldsymbol{x}\in\Gamma_2. \tag{20b}$$

In this system all boundary data are at the right-hand side of the equations.

Theorem 2 *If $C_l(\Gamma) = 1$ there exists a non-trivial pair of functions (q_1, u_2) such that the left-hand sides of (20a) and (20b) are equal to zero.*

Proof. We have to find a non-trivial pair of functions (q_1, u_2) such that the left-hand sides of (20a) and (20b) are equal to zero when $C_l(\Gamma) = 1$. Choose $u_2 \equiv 0$ and $q_1 = q_e|_{\Gamma_1} + h_1$, with the function h_1 satisfying

$$\mathcal{K}_1^s h_1 = \mathcal{K}_2^s q_e,\ \boldsymbol{x}\in\Gamma. \tag{21}$$

With these choices the left-hand sides of (20a) and (20b) are equal to

$$-\mathcal{K}_1^s q_1 = -(\mathcal{K}_1^s q_e + \mathcal{K}_1^s h_1) = -(\mathcal{K}_1^s q_e + \mathcal{K}_2^s q_e) = -\mathcal{K}^s q_e = \frac{1}{2\pi}\log C_l(\Gamma) = 0. \tag{22}$$

We still have to prove that it is possible to find a function h_1 that satisfies (21). First we note that the right-hand side of (21) is in $< q_e >^\perp$, since

$$\begin{aligned}(\mathcal{K}_2^s q_e, q_e)_\Gamma \quad &= \quad \int_\Gamma\int_{\Gamma_2} G(\boldsymbol{x},\boldsymbol{y})q_e(\boldsymbol{y})d\Gamma_y q_e(\boldsymbol{x})d\Gamma_x \\ &= \quad \int_{\Gamma_2}\int_{\Gamma} G(\boldsymbol{x},\boldsymbol{y})q_e(\boldsymbol{x})d\Gamma_x q_e(\boldsymbol{y})d\Gamma_y \\ &= \quad (\mathcal{K}^s q_e, q_e)_{\Gamma_2} = -\frac{1}{2\pi}\log C_l(\Gamma)(1, q_e)_{\Gamma_2} = 0.\end{aligned} \tag{23}$$

Here $(\cdot,\cdot)_\Gamma$ stands for the inner product over the boundary Γ. Therefore we can generalize the question: is it possible to find a function h_1 such that $\mathcal{K}_1^s h_1 = \phi$ for all $\phi \in < q_e >^\perp$? If so, then $\phi = \mathcal{K}_2^s q_e$ completes the proof.

For all functions $q \in < q_e >^\perp$ with $q \neq 0$ we have $I(q) > I(q_e)$, since q_e is the unique equilibrium distribution. Using $I(q) = 2\pi(\mathcal{K}^s q, q)$ we find that

$$(\mathcal{K}^s q, q) > \frac{1}{2\pi}I(q_e) = -\frac{1}{2\pi}\log C_l(\Gamma) = 0,\ q \in < q_e >^\perp,\ q \neq 0. \tag{24}$$

So $\mathcal{K}^s$ is positive definite and invertible on the function space $< q_e >^\perp$. This means that for all $\phi \in < q_e >^\perp$ there is a function h with $\mathcal{K}^s h = \phi$, namely $h = (\mathcal{K}^s)^\dagger \phi$, where † stands for the generalized inverse.

Let h_2 be a function at Γ_2, and let h be the composite function of h_1 and h_2,

$$h = \begin{cases} h_1, & \boldsymbol{x} \in \Gamma_1, \\ h_2, & \boldsymbol{x} \in \Gamma_2. \end{cases} \tag{25}$$

Recall that we search for a function h_1 such that $\mathcal{K}_1^s h_1 = \phi$, for $\phi \in < q_e >^\perp$. We add the function $\mathcal{K}_2^s h_2$ to this equation,

$$\mathcal{K}_1^s h_1 + \mathcal{K}_2^s h_2 = \phi + \mathcal{K}_2^s h_2, \tag{26}$$

which is equivalent to

$$\mathcal{K}^s h = \phi + \mathcal{K}_2^s h_2. \tag{27}$$

The right-hand side of this equation is in $< q_e >^\perp$, for $\phi \in < q_e >^\perp$ and

$$\begin{aligned} (\mathcal{K}_2^s h_2, q_e)_\Gamma &= \int_\Gamma \int_{\Gamma_2} G(\boldsymbol{x},\boldsymbol{y}) h_2(\boldsymbol{y}) d\Gamma_y q_e(\boldsymbol{x}) d\Gamma_x \\ &= \int_{\Gamma_2} \int_\Gamma G(\boldsymbol{x},\boldsymbol{y}) q_e(\boldsymbol{x}) d\Gamma_x h_2(\boldsymbol{y}) d\Gamma_y \\ &= (\mathcal{K}^s q_e, h_2)_{\Gamma_2} = -\frac{1}{2\pi} \log C_l(\Gamma)(1, h_2)_{\Gamma_2} = 0. \end{aligned} \tag{28}$$

Since $\mathcal{K}^s$ is invertible on the function space $< q_e >^\perp$ we find

$$h = (\mathcal{K}^s)^\dagger \left[\phi + \mathcal{K}_2^s h_2\right]. \tag{29}$$

The function h_1 is then the restriction of h to Γ_1, so

$$h_1 = \left((\mathcal{K}^s)^\dagger \left[\phi + \mathcal{K}_2^s h_2\right] \right)\Big|_{\Gamma_1}. \tag{30}$$

■

Theorem 2 tells us that the BIE for the mixed problem does not have a unique solution when $C_l(\Gamma) = 1$. Moreover, the division of Γ into a part Γ_1 with Dirichlet conditions and a part Γ_2 with Neuman conditions does not play a role in this. It does not make a difference whether we take Γ_1 very small or very large; the non-uniqueness behavior of the BIE relates solely to the whole boundary Γ.

4.3 Neuman problem

It is well known that the Neuman boundary value problem for the Laplace equation does not have a unique solution. For completeness we prove the following theorem for the BIE (4) with Neuman boundary conditions.

Theorem 3 *For any closed curve* Γ

$$(\frac{1}{2}\mathcal{I} + \mathcal{K}^d)1 = 0. \tag{31}$$

This implies that the Neuman problem has a solution which is unique up to a constant.

Proof. To show that operator $\frac{1}{2}\mathcal{I} + \mathcal{K}^d$ applied to the constant function 1 yields zero, we need to prove that $\mathcal{K}^d 1 \equiv -\frac{1}{2}$ at the boundary. Let $\boldsymbol{x}$ be a point at the boundary Γ, then using Gauss's theorem we find

$$\left(\mathcal{K}^d 1\right)(\boldsymbol{x}) = \int_\Gamma \frac{\partial}{\partial n_y} G(\boldsymbol{x},\boldsymbol{y}) d\Gamma_y = \int_\Omega \nabla_y^2 G(\boldsymbol{x},\boldsymbol{y}) d\Omega_y, \tag{32}$$

where the subscript y means integration or differentiation with respect to the variable $\boldsymbol{y}$. The fundamental solution is defined in such a way that $\nabla_y^2 G(\boldsymbol{x},\boldsymbol{y}) = 0$ in the interior of Ω. At the boundary however, we have to take special care, since the fundamental solution has a logarithmic singularity at the point $\boldsymbol{y} = \boldsymbol{x}$. Let B_ε be a small circle with radius ε around the point $\boldsymbol{x}$ and let B'_ε be the part of that circle that lies inside Ω, i.e. $B'_\varepsilon = B_\varepsilon \cap \Omega$, see Figure 1. The domain integral in (32) can be split in

$$\big(\mathcal{K}^d 1\big)(\boldsymbol{x}) \quad = \quad \int_{\Omega/B'_\varepsilon} \nabla_y^2 G(\boldsymbol{x},\boldsymbol{y}) d\Omega_y + \int_{B'_\varepsilon} \nabla_y^2 G(\boldsymbol{x},\boldsymbol{y}) d\Omega_y. \tag{33}$$

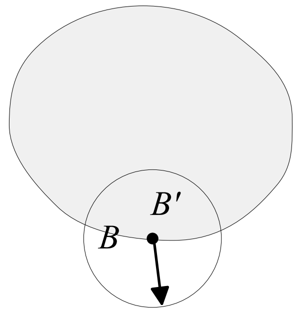


Figure 1: *The point $\boldsymbol{x} \in \Gamma$ is the center of a small circle B_ε with radius ε.*

Within the domain Ω/B'_ε the fundamental solution does not have a singular point and thus $\nabla_y^2 G(\boldsymbol{x},\boldsymbol{y}) = 0$ in this domain. As a consequence the first integral at the right-hand side of (33) is equal to zero. If the boundary Γ is smooth enough, the circle B'_ε is half the size of the circle B_ε. Likewise, if ε goes to zero, the integral over B'_ε in (33) is half the size of the same integral over B_ε. Hence we obtain

$$\big(\mathcal{K}^d 1\big)(\boldsymbol{x}) \quad = \quad \frac{1}{2} \int_{B_\varepsilon} \nabla_y^2 G(\boldsymbol{x},\boldsymbol{y}) d\Omega_y. \tag{34}$$

We use Gauss's theorem to go to a boundary integral,

$$\big(\mathcal{K}^d 1\big)(\boldsymbol{x}) \quad = \quad \frac{1}{2} \int_{\Gamma_\varepsilon} \frac{\partial}{\partial n_y} G(\boldsymbol{x},\boldsymbol{y}) d\Gamma_y, \tag{35}$$

where Γ_ε is the boundary of the circle B_ε. We introduce polar coordinates (r,θ) on the circle B_ε, the point $\boldsymbol{x}$ being the local origin. Recall the definition of the fundamental solution (2) in which now $\|\boldsymbol{x} - \boldsymbol{y}\| = r$ for $\boldsymbol{y} \in B_\varepsilon$. It is straightforward to see that

$$\frac{\partial}{\partial n_y} G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2\pi} \frac{\partial}{\partial r} \log \frac{1}{r} = -\frac{1}{2\pi} \frac{1}{r}. \tag{36}$$

Substituting this in the integral of (35) results in

$$\big(\mathcal{K}^d 1\big)(\boldsymbol{x}) \quad = \quad -\frac{1}{2} \int_0^{2\pi} \frac{1}{2\pi} \frac{1}{r}\, r d\theta = -\frac{1}{2}. \tag{37}$$

The direct consequence of this is that

$$\big(\frac{1}{2}\mathcal{I} + \mathcal{K}^d\big) 1 = 0, \tag{38}$$

with which Theorem 3 has been proven. ■

5 Examples

In this section we illustrate the results from Section 4. We do this by calculating the condition number of the matrices that appear in the BEM. After discretization of the domain the BIE transforms into a linear system of equations. If the BIE is not uniquely solvable the condition number of the corresponding system matrix goes to infinity. The boundary curve Γ is divided into N equispaced linear elements Γ_l, $l = 1, \ldots, N$. At each element the functions u and q are approximated by a constant value, i.e. $u \approx u_l$ and $q \approx q_l$ on Γ_l, $l = 1, \ldots, N$.

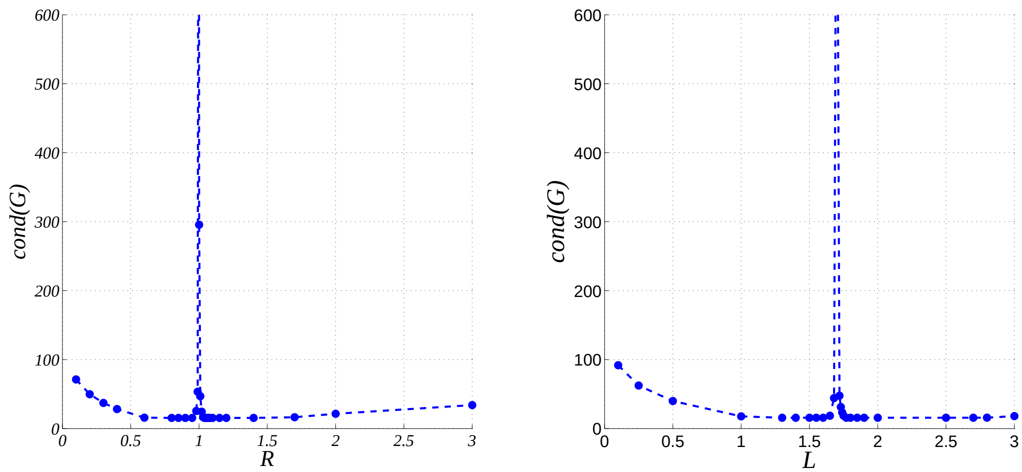
5.1 Examples Dirichlet problem

For the BIE related to the Laplace equation with Dirichlet boundary equations we obtain the following linear system

$$\mathbf{G}\mathbf{q} = \mathbf{f}, \tag{39}$$

where $\mathbf{f} = \mathbf{f}(\tilde{u})$. The vector $\mathbf{q}$ contains the coefficients q_i, and the matrix elements of $\mathbf{G}$ are calculated with

$$G_{lk} = \int_{\Gamma_k} G(\boldsymbol{x}_l, \boldsymbol{y}) d\Gamma_y,\ l, k = 1, \ldots, N. \tag{40}$$



(a) Circle with radius R

(b) Square with side L

Figure 2: *The condition number of* $\mathbf{G}$ *for the Laplace problem with Dirichlet boundary conditions.*

If $l \neq k$ the matrix elements are approximated by using Gauss quadrature rule. For $l = k$ it can be shown [2] that

$$G_{ll} = \frac{|\Gamma_l|}{2\pi}\left(1 + \log\frac{2}{|\Gamma_l|}\right), \tag{41}$$

where $|\Gamma_l|$ is the length of the l-th element. We perform these calculations for two cases: a circular domain with radius R and a square domain with side L. In both cases we choose $N = 36$.

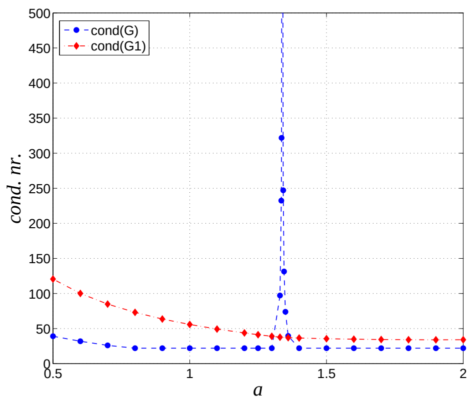
For the circular domain the logarithmic capacity is equal to the radius of the circle, see Table 1. Thus if the radius R is equal to one also the logarithmic capacity is equal to one. In that case the BIE does not have a unique solution and the condition number of the matrix $\mathbf{G}$ will be very large. In Figure 2(a) we show the condition number of $\mathbf{G}$ as a function of the radius R. We indeed observe that the condition number goes to infinity when R is equal to one. Similar results have been observed in [4], [3]. More details about estimating the condition number can be found in [5].

For the square domain the logarithmic capacity is approximately $0.59L$, see Table 1. Hence if the length L of the side is approximately equal to $L^* := 1/0.59 = 1.69$, then the logarithmic capacity is equal to one. Analogous to the case of the circle the condition number of the matrix $\mathbf{G}$ is very large in that case. In Figure 2(b) the condition number of $\mathbf{G}$ is plotted as a function of L. We observe that it is going to infinity when L is close to 1.69.

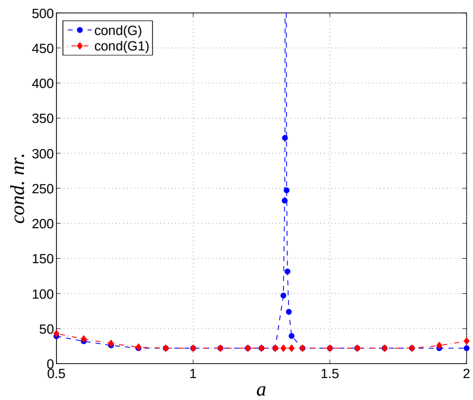
It is clear that we cannot use the boundary element formulation in the case that the logarithmic capacity is equal to one. In Section 4.1 we suggested to search for solutions q of the Dirichlet BIE that have contour integral equal to zero. Translating this condition to the boundary element formulation we have to search for solutions $\mathbf{q}$ that satisfy the condition $q_1 + \ldots + q_N = 0$. The authors in [17] describe a procedure to add this equation to the existing linear system in such a way that a new square system is obtained. They propose to solve the system

$$\underbrace{\left[\begin{array}{ccc|c} & & & K \\ & \mathbf{G} & & \vdots \\ & & & K \\ \hline H & \ldots & H & 0 \end{array}\right]}_{\mathbf{G}_1} \begin{bmatrix} q_1 \\ \vdots \\ q_N \\ w \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \\ 0 \end{bmatrix}, \tag{42}$$

where w is an additional unknown, and K and H are scalars that can be chosen arbitrarily. It is shown that when both K and H are of order $N^{-1/2}$ the condition number of the new system matrix $\mathbf{G}_1$ is minimal, that is of order N.



(a) $H = K = N^{-1/2}$



(b) $H = K = 0.13N^{-1/2}$

Figure 3: *The condition number of the matrices $\mathbf{G}$ (circles) and $\mathbf{G}_1$ (diamonds) for the Laplace problem with mixed boundary conditions on an ellipsoidal domain.*

We illustrate this for an ellipsoidal domain with axis a and $a/2$. In Table 1 is given that the logarithmic capacity of such an ellipse is equal to $3a/4$. Hence if we choose a equal to the value $a^* = 4/3$ the logarithmic capacity is equal to one. In Figure 3(a) both the condition numbers of $\mathbf{G}$ (dashed) and $\mathbf{G}_1$ (solid) are plotted as a function of the scaling parameter a. We observe that for $a = a^*$ the condition number of $\mathbf{G}$ is going to infinity. The condition number of $\mathbf{G}_1$ remains bounded. Note that when the logarithmic capacity is not equal to one the condition number of $\mathbf{G}_1$ is larger than the condition number of $\mathbf{G}$. The conditioning of the matrix $\mathbf{G}_1$ can be improved further by varying the scalars K and H. It turns out that, for this specific case, that $K = H = 0.13N^{-1/2}$ yields a minimal condition number for $\mathbf{G}_1$, see Figure 3(b).

5.2 Examples mixed problem

After discretization of the domain the BIE for the Laplace equation with mixed boundary conditions transforms in the linear system

$$\mathbf{G}\mathbf{q} = \left(\frac{1}{2}\mathbf{I} + \mathbf{H}\right)\mathbf{u}. \tag{43}$$

The matrix $\mathbf{G}$ is the same matrix as for the Dirichlet case, and the elements of the matrix $\mathbf{H}$ are calculated with

$$H_{lk} = \int_{\Gamma_k} \frac{\partial}{\partial n_y} G(\boldsymbol{x}_l, \boldsymbol{y}) d\Gamma_y = \frac{1}{2\pi} \int_{\Gamma_k} \frac{(\boldsymbol{x}_l - \boldsymbol{y}, \boldsymbol{n}_y)}{\|\boldsymbol{x}_l - \boldsymbol{y}\|^2} \, d\Gamma_y, \; l, k = 1, \ldots, N. \tag{44}$$

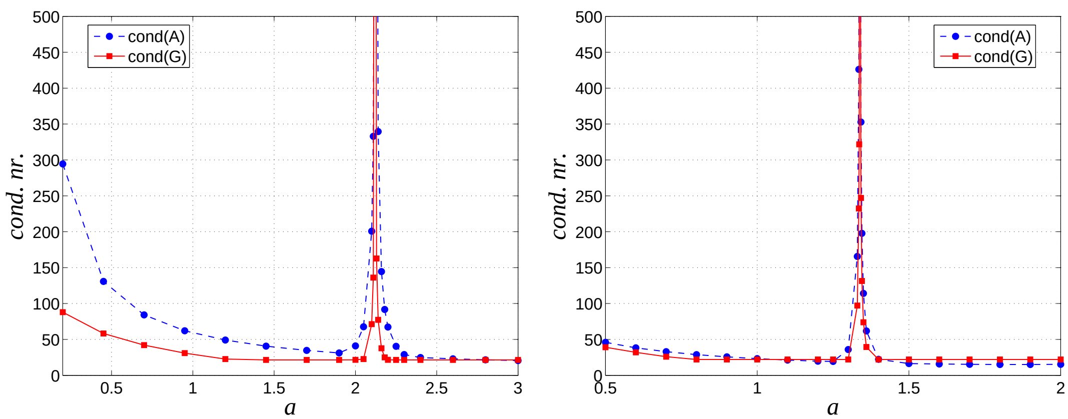
It can be shown that the diagonal elements of $\mathbf{H}$ are equal to zero. The off-diagonal elements are calculated with a Gauss quadrature rule. We assume that on the first part of the boundary Γ, represented by the first m ($0 \leq m \leq N$) elements, Dirichlet boundary conditions are given. On the remaining $N-m$ elements we have Neuman boundary conditions. This implies that the first m coefficients of $\mathbf{u}$ and the last $N-m$ coefficients of $\mathbf{q}$ are given. By moving all unknown coefficients to the left-hand side and all known coefficients to the right-hand side in (43) we arrive at the standard form linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{45}$$

If the condition number of this matrix $\mathbf{A}$ goes to infinity then the BIE is not uniquely solvable.

We give two examples: a triangular domain and a ellipsoidal domain. The triangle-like domain is an isosceles right triangle with sides of length a. For such a triangle the logarithmic capacity is given by

$$C_l(\text{triangle}) = \frac{3^{3/4}\Gamma^2(1/4)}{2^{7/2}\pi^{3/2}}\, a \approx 0.476\, a. \tag{46}$$



(a) Triangle

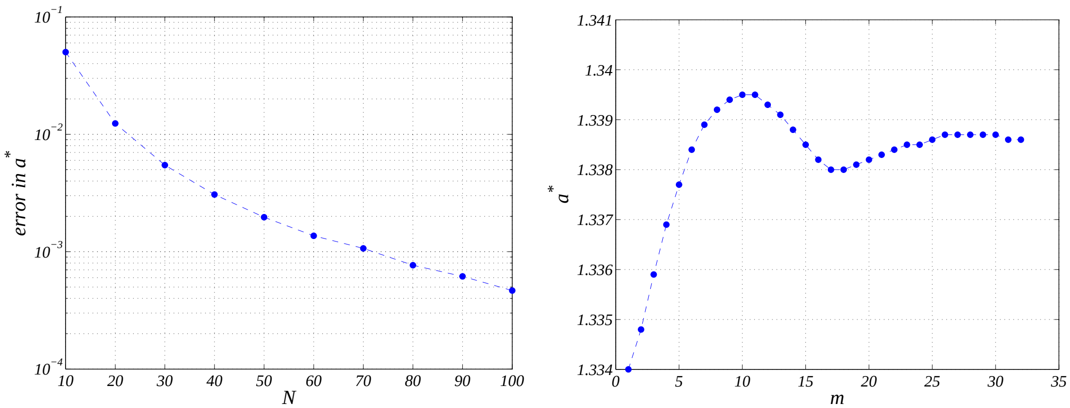
(b) Ellipse

Figure 4: *The condition number of the matrices* $\mathbf{A}$ *(dashed) and* $\mathbf{G}$ *(solid), which relate to the Laplace problem with mixed boundary conditions and Dirichlet boundary conditions respectively.*

This implies that the condition number will be large when the scaling parameter a is close to $a^* := 1/0.476 \approx 2.1$. The ellipse has semi-axes of length a and $a/2$. Therefore the logarithmic capacity of the ellipse-like domain is $3a/4$. Hence we may expect a large condition number when the scaling parameter a is close to $a^* := 4/3$.

In Figure 4 we show the condition numbers for the matrices $\mathbf{A}$ and $\mathbf{G}$. The dashed lines represent the condition number for the mixed problem, while the solid line represents the condition number for the Dirichlet problem. For both domains we observe that the location of the scaling parameter a^* is the same for both matrices $\mathbf{A}$ and $\mathbf{G}$. The small difference that is present is caused by numerical inaccuracies due to the Gauss quadrature. As was predicted for the triangle, the point where the condition number is very large is close to $a \approx 2.1$. For the ellipse we observe that indeed the point where the condition number is large is at $a \approx 1.3$.

Figure 5 gives more details about the critical scaling parameter for which the condition number goes to infinity. Again we consider a ellipsoidal domain with semi-axes of length a and $a/2$. The logarithmic capacity is equal to $3a/4$, so we expect to find a critical scaling parameter $a^* = 4/3$. This, of course, holds for a perfect ellipse. In reality we work with an approximation of an ellipse, namely a polygon with N sides. If N is large the domain is close to the ellipse, and we expect to find a scaling parameter a^* that is close to that of the ellipse, i.e. $a^* = 4/3$. In Figure 5(a) we see the accuracy in a^* as a function of N. We observe that for large N the error between theoretical value and actual value gets very small. In Section 4.2 it was already mentioned that the division of the boundary Γ into a Dirichlet and a Neuman part does not play a role in the uniqueness properties of the boundary integral equation. Figure 5(b) illustrates this. Here we vary m, the number of elements that have Dirichlet boundary conditions. The total number of elements is $N = 32$. Hence $m = 32$ corresponds to the Dirichlet problem, while $m = 1$ is a problem with Neuman conditions, except for one element. For each value of m we find the scaling parameter a^*. We see that there is little change in the value of a^* as m varies between 1 and N.



(a) The error in a^* as a function of N.

(b) The value of a^* as a function of m ($N = 32$).

Figure 5: *The critical scaling parameter a^* for which the condition number of $\mathbf{A}$ goes to infinity.*

As the mixed problem is ill-posed when the logarithmic capacity equals one, we need to add an extra condition like we did for the Dirichlet problem. Since u satisfies the Laplace equation on the interior of the domain we know that q must have a zero contour integral. This leads to the following condition for the solution vector $\mathbf{q}$ of the linear system in (43),

$$q_1 + \ldots + q_m = -\gamma := -(\tilde{q}_{m+1} + \ldots + \tilde{q}_N), \tag{47}$$

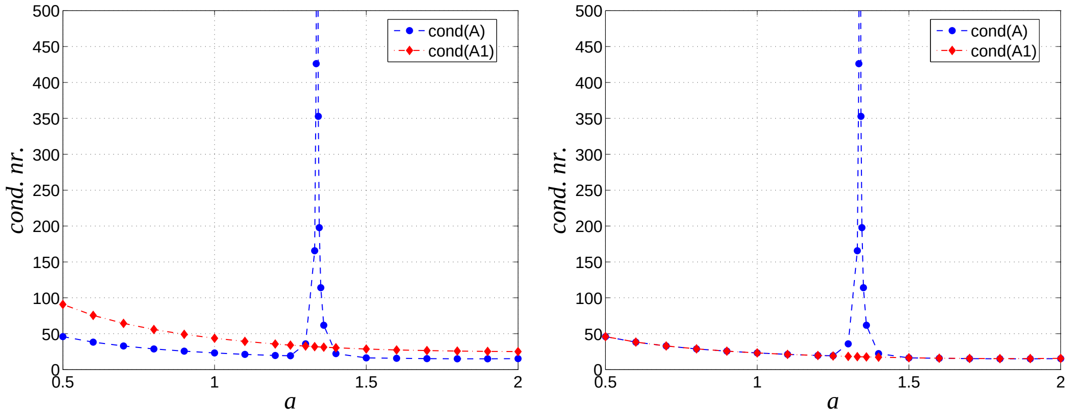
since part of the vector $\mathbf{q}$ is already prescribed by the boundary condition at Γ_2. Like we did for the Dirichlet case we formulate a new linear system in which the extra condition is incorporated,

$$\underbrace{\left[\begin{array}{cc|c} & & K \\ \multicolumn{2}{c|}{\mathbf{A}} & \vdots \\ & & K \\ \hline H \ldots H & 0 \ldots 0 & 0 \end{array}\right]}_{\mathbf{A}_1} \begin{bmatrix} x_1 \\ \vdots \\ x_N \\ w \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \\ -H\gamma \end{bmatrix}. \tag{48}$$

Again w is an additional unknown and K and H are scalars of order $N^{-1/2}$. To investigate the condition number of the new matrix $\mathbf{A}_1$ we use the same example as in the previous section. For the ellipse with axis a and $a/2$ the logarithmic capacity is equal to $3a/4$. Hence for a equal to $a^* := 4/3$ the logarithmic capacity is equal to one. In Figure 6(a) we give the condition number of the matrices $\mathbf{A}$ (dashed) and $\mathbf{A}_1$ (solid) as a function of the scaling parameter a. We observe that for $a = a^*$ the condition number of $\mathbf{A}$ goes to infinity, while the condition number of $\mathbf{A}_1$ remains bounded. However, when $C_l(\Gamma) \neq 1$ the matrix $\mathbf{A}_1$ has a slightly larger condition number than the matrix $\mathbf{G}$. The conditioning can be improved by changing the parameters K and H. For this particular case we choose $K = H = 0.2N^{-1/2}$, which yields a minimal condition number of the matrix $\mathbf{A}_1$ (see Figure 6(b)).

6 Conclusion

We have shown that for the BIE for the Laplace problem with Dirichlet boundary conditions uniqueness of the solution cannot always be guaranteed. Namely, if the size of the domain is such that the logarithmic



(a) $H = K = N^{-1/2}$ (b) $H = K = 0.2N^{-1/2}$

Figure 6: *The condition number of the matrices* $\mathbf{A}$ *(circles) and* $\mathbf{A}_1$ *(diamonds) for the Laplace problem with mixed boundary conditions on an ellipsoidal domain.*

capacity of the boundary is equal to one, the corresponding integral operator $\mathcal{K}^s$ has an eigenfunction with eigenvalue zero. We have illustrated this by computing the condition number of the BEM-matrix $\mathbf{G}$ related to the integral operator. This matrix is ill-conditioned when the logarithmic capacity is equal to one.

One way to avoid the non-uniqueness problem is to rescale the domain such that its Euclidean diameter is smaller than one. Then the logarithmic capacity will also be smaller than one and a unique solution of the BIE does exist. As a consequence the matrix $\mathbf{G}$ is well-conditioned. Another option is to restrict the function space for the solution q by adding the constraint that the contour integral of q is equal to zero. The BIE supplemented with this equation has a unique solution. For the BEM formulation we also add an extra equation; the sum of coefficients of the solution vector $\mathbf{q}$ is equal to zero. The condition number of the resulting new matrix remains bounded.

The BIE for the Laplace problem with mixed boundary conditions is also not uniquely solvable when the logarithmic capacity of the boundary is equal to one. Remarkably, the non-uniqueness behavior is determined by the logarithmic capacity of the whole boundary, and not by the logarithmic capacity of the part of the boundary on which Dirichlet conditions are posed. This implies that the division of the boundary into Dirichlet and Neuman part does not influence the uniqueness properties of the BIE. Again we have the same two options as for the Dirichlet problem to assure a unique solution.

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