面積分轉線積分

回顧以前學過的

$$A = \int_A dA$$ 面積

$$\bar{y} = \frac{\int_A y \; dA}{\int_A dA}$$ 形心

$$I_{yy} = \int x^2 dA$$ 慣性矩

格林第二恆等式

$$\int_A \nabla u \cdot \nabla v dA + \int_A u \nabla^2 v dA = \int_\Gamma u \frac{\partial v}{\partial n} d\Gamma$$

$$\int_A \nabla \cdot \underset{\sim}{v}\, dA = \int_\Gamma \underset{\sim}{v} \cdot \underset{\sim}{n}\, d\Gamma$$ ⟸ 高斯定理

$$Let: \underset{\sim}{v} = \phi \nabla \varphi \;\; ; \phi = \phi(x,y) \;\; ; \varphi = \varphi(x,y)$$

$$\nabla \cdot (\phi \nabla \varphi) = \nabla \phi \cdot \nabla \varphi + \phi \nabla \cdot \nabla \varphi = \nabla \phi \cdot \nabla \varphi + \phi \nabla^2 \varphi$$

- $\nabla \Rightarrow$ 梯度算子
- $\phi(x,y) \Rightarrow$ 純量函數
- $\varphi(x,y) \Rightarrow$ 純量函數
- $\nabla \varphi \Rightarrow \varphi$ 的梯度

$$\int_A \nabla \phi \cdot \nabla \varphi + \phi \nabla^2 \varphi \, dA = \int_\Gamma \phi \nabla \varphi \cdot \underset{\sim}{n}\, d\Gamma$$

計算面積$\int_A 1\,dA \Rightarrow$ 轉線積分＝？

$$\int_A \phi\nabla^2\varphi\, dA = \int_\Gamma \phi\nabla\varphi\cdot \underset{\sim}{n}\, d\Gamma - \int_A \nabla\phi\cdot\nabla\varphi dA$$

$$\int_A 1\, dA \Rightarrow \int_A \phi\nabla^2\varphi\, dA$$

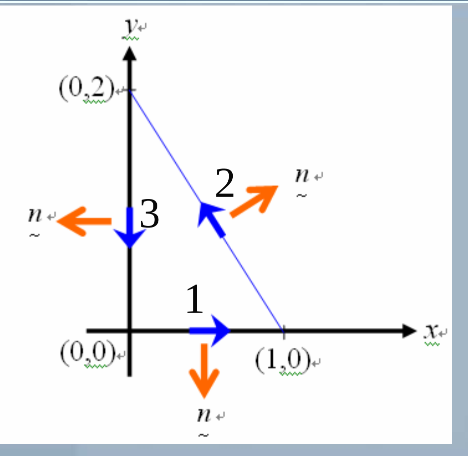
Let:$\phi = 1\ ,\varphi = y^2$

$$\Rightarrow \int_A 1\, dA = \frac{1}{2}\int_A \nabla^2 y^2\, dA$$

$$\int_A \nabla^2 y^2\, dA = \int_\Gamma \nabla y^2\cdot \underset{\sim}{n}\, d\Gamma - \int_A \nabla 1\cdot\nabla y^2 dA = \int_\Gamma \nabla y^2\cdot \underset{\sim}{n}\, d\Gamma$$

$$\Rightarrow \int_A 1\, dA = \frac{1}{2}\int_\Gamma \nabla y^2\cdot \underset{\sim}{n}\, d\Gamma$$

例題1.計算面積



$$y=2-2x,\ x=\frac{2-y}{2}$$

$$\left(\frac{d\Gamma}{dy}\right)^2=\left(\frac{dx}{dy}\right)^2+\left(\frac{dy}{dy}\right)^2=\left(\frac{d}{dy}\left(\frac{2-y}{2}\right)\right)^2+\left(\frac{dy}{dy}\right)^2$$

$$\frac{d\Gamma}{dy}=\frac{\sqrt{5}}{2}\ \Rightarrow\ d\Gamma=\frac{\sqrt{5}}{2}dy$$

$$A=\frac{1}{2}\int_\Gamma \nabla y^2\cdot \underset{\sim}{n}\,d\Gamma$$

$y=0$

$$=\int_{\Gamma_1}(0,y)\cdot(0,-1)d\Gamma+\int_{\Gamma_2}(0,y)\cdot\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)d\Gamma+\int_{\Gamma_3}(0,y)\cdot(-1,0)d\Gamma$$

$$=\int_0^2\frac{1}{\sqrt{5}}y\cdot\frac{\sqrt{5}}{2}dy=\frac{1}{4}y^2\Big|_0^2=1$$

推導通式 $\iint_A y^n\,dxdy = \int_A y^n\,dA$ 轉線積分＝？

$$\int_A \nabla\phi\cdot\nabla\varphi dA = \int_\Gamma \phi\nabla\varphi\cdot\underset{\sim}{n}\,d\Gamma - \int_A \phi\nabla^2\varphi\,dA$$

$$\int_A y^n\,dA \Rightarrow \int_A \nabla\phi\cdot\nabla\varphi dA$$

$$\text{Let:}\phi = y^2 \quad ,\varphi = y^n$$

$$\int_A y^n\,dA = \frac{1}{2n}\int_A \nabla y^2\cdot\nabla y^n dA$$

$$\int_A \nabla y^2\cdot\nabla y^n dA = \int_\Gamma y^2\nabla y^n\cdot\underset{\sim}{n}\,d\Gamma - \int_A y^2\nabla^2 y^n\,dA$$

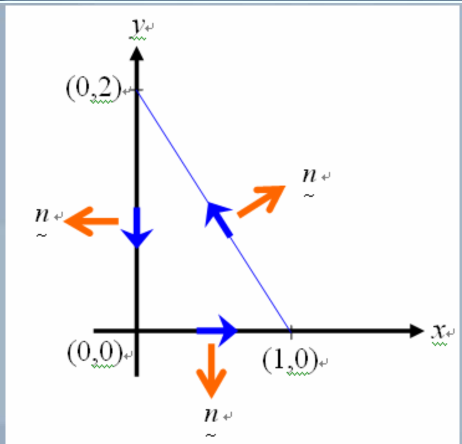
$$\Rightarrow \int_A y^n\,dA = \frac{1}{2n}\left[\int_\Gamma y^2\nabla y^n\cdot\underset{\sim}{n}\,d\Gamma - \int_A y^2\nabla^2 y^n\,dA\right]$$

$$2n\int_A y^n\,dA = \int_\Gamma y^2\nabla y^n\cdot\underset{\sim}{n}\,d\Gamma - \left(n\left(n-1\right)\right)\int_A y^n\,dA$$

$$\left(n^2 - n + 2n\right)\int_A y^n\,dA = \int_\Gamma y^2\nabla y^n\cdot\underset{\sim}{n}\,d\Gamma$$

$$\Rightarrow \int_A y^n\,dA = \frac{1}{n\left(n+1\right)}\int_\Gamma y^2\nabla y^n\cdot\underset{\sim}{n}\,d\Gamma$$

例題2.計算形心 n=1



$$y = 2 - 2x,\ x = \frac{2-y}{2}$$

$$\left(\frac{d\Gamma}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2 = \left(\frac{d}{dy}\left(\frac{2-y}{2}\right)\right)^2 + \left(\frac{dy}{dy}\right)^2$$

$$\frac{d\Gamma}{dy} = \frac{\sqrt{5}}{2} \quad \Rightarrow d\Gamma = \frac{\sqrt{5}}{2}\,dy$$

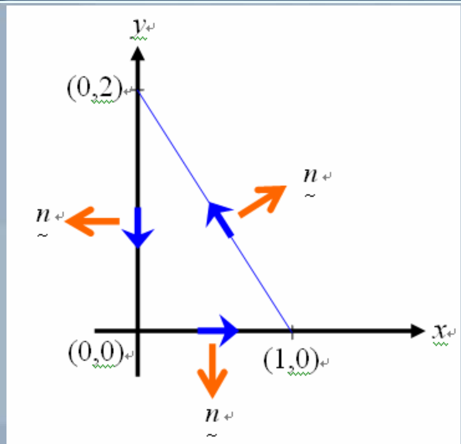
$$A\bar{y} = 1\cdot\bar{y} = \int_A y\,dA = \frac{1}{2}\int_\Gamma y^2 \nabla y \cdot \underset{\sim}{n}\,d\Gamma$$

$y = 0$

$$= \frac{1}{2}\int_{\Gamma_1} (0, y^2)\cdot(0,-1)\,d\Gamma + \frac{1}{2}\int_{\Gamma_2} (0, y^2)\cdot\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)d\Gamma + \frac{1}{2}\int_{\Gamma_3} (0, y^2)\cdot(-1,0)\,d\Gamma$$

$$= \frac{1}{2}\int_0^2 \frac{1}{\sqrt{5}} y^2 \cdot \frac{\sqrt{5}}{2}\,dy = \frac{1}{2}\cdot\left(\left.\frac{y^3}{6}\right|_0^2\right) = \frac{2}{3}$$

例題3.計算慣性矩　n=2



$$y = 2-2x,\; x = \frac{2-y}{2}$$

$$\left(\frac{d\Gamma}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2 = \left(\frac{d}{dy}\left(\frac{2-y}{2}\right)\right)^2 + \left(\frac{dy}{dy}\right)^2$$

$$\frac{d\Gamma}{dy} = \frac{\sqrt{5}}{2} \Rightarrow d\Gamma = \frac{\sqrt{5}}{2}\,dy$$

$$I_{xx} = \int_A y^2\, dA = \frac{1}{6}\int_\Gamma y^2 \nabla y^2 \cdot \underset{\sim}{n}\, d\Gamma$$

$y = 0$

$$= \frac{1}{3}\int_{\Gamma_1} (0, y^3)\cdot(0,-1)\, d\Gamma + \frac{1}{3}\int_{\Gamma_2} (0, y^3)\cdot\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) d\Gamma + \frac{1}{3}\int_{\Gamma_3} (0, y^3)\cdot(-1, 0)\, d\Gamma$$

$$= \frac{1}{3}\int_0^2 \frac{1}{\sqrt{5}} y^3 \cdot \frac{\sqrt{5}}{2}\, dy = \frac{1}{3}\cdot\left(\left.\frac{y^4}{8}\right|_0^2\right) = \frac{2}{3}$$

推導通式 $\iint\limits_{A} x^n y^m \, dxdy = \int\limits_{A} x^n y^m \, dA$ 轉線積分＝？

$$\int\limits_{A} \nabla\phi \cdot \nabla\varphi dA = \int_{\Gamma} \phi \nabla\varphi \cdot \underset{\sim}{n} \, d\Gamma - \int\limits_{A} \phi \nabla^2 \varphi \, dA$$

$$\int\limits_{A} x^n y^m \, dA \Rightarrow \int\limits_{A} \nabla\phi \cdot \nabla\varphi dA$$

$$\text{Let:} \phi = x^n y^2 \quad , \varphi = y^m$$

$$\int\limits_{A} x^n y^m \, dA = \frac{1}{2m} \int\limits_{A} \nabla x^n y^2 \cdot \nabla y^m dA$$

$$\int\limits_{A} \nabla x^n y^2 \cdot \nabla y^m dA = \int_{\Gamma} x^n y^2 \nabla y^m \cdot \underset{\sim}{n} \, d\Gamma - \int\limits_{A} x^n y^2 \nabla^2 y^m \, dA$$

$$\Rightarrow \int\limits_{A} x^n y^m \, dA = \frac{1}{2m} \left[\int_{\Gamma} x^n y^2 \nabla y^m \cdot \underset{\sim}{n} \, d\Gamma - \int\limits_{A} x^n y^2 \nabla^2 y^m \, dA \right]$$

$$2m \int\limits_{A} x^n y^m \, dA = \int_{\Gamma} x^n y^2 \nabla y^m \cdot \underset{\sim}{n} \, d\Gamma - \left(m(m-1) \right) \int\limits_{A} \left(x^n y^2 \right) y^{m-2} \, dA$$

$$(m^2 + m) \int\limits_{A} x^n y^m \, dA = \int_{\Gamma} x^n y^2 \nabla y^m \cdot \underset{\sim}{n} \, d\Gamma$$

$$\int\limits_{A} x^n y^m \, dA = \frac{1}{m(m+1)} \int_{\Gamma} x^n y^2 \nabla y^m \cdot \underset{\sim}{n} \, d\Gamma$$