Green's third identity

$$2\pi G(x,\xi)=\int_B T(s,x)G(s,\xi)dB(s)-\int_B U(s,x)\frac{\partial G(s,\xi)}{\partial n_s}dB(s)+U(\xi,x),\quad x\in D\cup B$$

$$0=\int_B T(s,x)G(s,\xi)dB(s)-\int_B U(s,x)\frac{\partial G(s,\xi)}{\partial n_s}dB(s)+U(\xi,x),\quad x\in D^c\cup B$$

$$U(s,x)=\begin{cases}\ln R-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R})^m\cos m(\theta-\phi), & R\ge\rho\\ \ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{R}{\rho})^m\cos m(\theta-\phi), & R<\rho\end{cases}$$

Solution:

Null-field integral equation

$$0=\int_B T(s,x)G(s,\xi)dB(s)-\int_B U(s,x)\frac{\partial G(s,\xi)}{\partial n_s}dB(s)+U(\xi,x),\quad x\in D^c\cup B$$

known $\frac{\partial G(s,\xi)}{\partial n_s}\Rightarrow t=0$, unknown $G(s,\xi)=P_0+\sum_{n=1}^{\infty}P_n\cos n\theta+Q_n\sin n\theta$

$$T(s,x)=\begin{cases}T^i(R,\theta;\rho,\phi)=\frac{1}{R}+\sum_{m=1}^{\infty}(\frac{\rho^m}{R^{m+1}})\cos m(\theta-\phi), & R>\rho\\ T^e(R,\theta;\rho,\phi)=-\sum_{m=1}^{\infty}(\frac{R^{m-1}}{\rho^m})\cos m(\theta-\phi), & \rho>R\end{cases}$$

$$0=-\int_B T(s,x)G(s,\xi)dB(s)+U(\xi,x),\quad x\in D^c\cup B$$

$$0=-\int_{-\pi}^{\pi}\left[\frac{1}{R}+\sum_{m=1}^{\infty}(\frac{\rho^m}{R^{m+1}})\cos m(\theta-\phi)\right]\left[P_0+\sum_{n=1}^{\infty}P_n\cos n\theta+Q_n\sin n\theta\right]Rd\theta$$
$$+\left[\ln R_\xi-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R_\xi})^m\cos m(\theta_\xi-\phi)\right],\quad x\in D^c\cup B$$

$$0=-\left[2\pi P_0+\sum_{m=1}^{\infty}R(\frac{\rho^m}{R^{m+1}})(P_n\pi\cos m\phi+Q_n\pi\sin m\phi)\right]$$
$$+\left[\ln R_\xi-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R_\xi})^m\cos m(\theta_\xi-\phi)\right],\quad x\in D^c\cup B$$

$$\left[2\pi P_0+\sum_{m=1}^{\infty}R(\frac{\rho^m}{R^{m+1}})(P_n\pi\cos m\phi+Q_n\pi\sin m\phi)\right]=\left[\ln R_\xi-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R_\xi})^m\cos m(\theta_\xi-\phi)\right]$$

$$\left[2\pi P_0+\sum_{m=1}^{\infty}R(\frac{\rho^m}{R^{m+1}})(P_n\pi\cos m\phi+Q_n\pi\sin m\phi)\right]=\left[\ln R_\xi-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R_\xi})^m\cos m\theta_\xi\cos m\phi+\sin m\theta_\xi\sin m\phi\right]$$

$$R=\rho=a$$

$$P_0 = \frac{\ln R_\xi}{2\pi} \;\; ; P_n = -\frac{1}{\pi m}\left(\frac{a}{R_\xi}\right)^m \cos m\theta_\xi \;\; ; Q_n = -\frac{1}{\pi m}\left(\frac{a}{R_\xi}\right)^m \sin m\theta_\xi$$

$$\frac{\partial G(s,\xi)}{\partial n_s} = P_0 + \sum_{n=1}^{\infty} P_n \cos n\theta + Q_n \sin n\theta = \frac{\ln R_\xi}{2\pi} - \sum_{m=1}^{\infty} \frac{1}{\pi m}\left(\frac{a}{R_\xi}\right)^m \cos m\theta_\xi + - \frac{1}{\pi m}\left(\frac{a}{R_\xi}\right)^m \sin m\theta_\xi$$

$$2\pi G(x,\xi) = \int_B T(s,x)G(s,\xi)dB(s) - \int_B U(s,x)\frac{\partial G(s,\xi)}{\partial n_s}dB(s) + U(\xi,x), \quad x \in D \cup B$$

$$2\pi G(x,\xi) = \left[\ln R_\xi - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho}{R_\xi}\right)^m \cos m(\theta_\xi - \phi)\right] +$$

$$\int_{-\pi}^{\pi}\left[-\sum_{m=1}^{\infty}\left(\frac{R^{m-1}}{\rho^m}\right)(\theta - \phi)\right]\left[\frac{\ln R_\xi}{2\pi} - \sum_{m=1}^{\infty}\frac{1}{\pi m}\left(\frac{a}{R_\xi}\right)^m \cos m\theta_\xi \cos m\theta + \frac{1}{\pi m}\left(\frac{a}{R_\xi}\right)^m \sin m\theta_\xi \sin m\theta\right] R d\theta$$

$$G(x,\xi) = \frac{1}{2\pi}\left[\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R^m}{\rho^m}\right)\left(\frac{a}{R_\xi}\right)^m \left(\cos m\phi \cos m\theta_\xi + \sin m\phi \sin m\theta_\xi\right)\right]$$

$$+\left[\ln R_\xi - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho}{R_\xi}\right)^m \cos m(\theta_\xi - \phi)\right] \qquad , R_\xi \ge \rho$$

$$2\pi G(x,\xi) = \left[\ln \rho - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R_\xi}{\rho}\right)^m \cos m(\theta - \phi)\right] +$$

$$\int_{-\pi}^{\pi}\left[-\sum_{m=1}^{\infty}\left(\frac{R^{m-1}}{\rho^m}\right)(\theta - \phi)\right]\left[\frac{\ln a}{2\pi} - \sum_{m=1}^{\infty}\frac{1}{\pi m}\left(\frac{a}{R_\xi}\right)^m \cos m\theta_\xi \cos m\theta + \frac{1}{\pi m}\left(\frac{a}{R_\xi}\right)^m \cos m\theta_\xi \cos m\theta\right] R d\theta$$

$$G(x,\xi) = \frac{1}{2\pi}\left[\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R^m}{\rho^m}\right)\left(\frac{a}{R_\xi}\right)^m \left(\cos m\phi \cos m\theta_\xi + \sin m\phi \sin m\theta_\xi\right)\right]$$

$$+\left[\ln \rho - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R_\xi}{\rho}\right)^m \cos m(\theta_\xi - \phi)\right] \qquad , R_\xi < \rho$$

Superposition

Fundamental solution

$$2\pi u_1(x)=U(s,x)$$

$$U(s,x)=\begin{cases}\ln R-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R})^m\cos m(\theta-\phi), & R\geq\rho\\ \ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{R}{\rho})^m\cos m(\theta-\phi), & R<\rho\end{cases}$$

BIE for the BVP

$$2\pi u_2(x)=\int_B T(s,x)G(s,\xi)dB(s)-\int_B U(s,x)\frac{\partial G(s,\xi)}{\partial n_s}dB(s)$$

$$u_1+u_2=G(x,\xi)$$

Solution:

Present method:

$$2\pi u_1(x)=U(s,x)$$

$$u_1=\begin{cases}\frac{1}{2\pi}\left[\ln\bar{R}-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{\bar{R}})^m\cos m(\bar{\theta}-\phi)\right], & R\geq\rho\\ \frac{1}{2\pi}\left[\ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\bar{R}}{\rho})^m\cos m(\bar{\theta}-\phi)\right], & R<\rho\end{cases}$$

$$T(s,x)=\begin{cases}T^i(R,\theta;\rho,\phi)=\frac{1}{R}+\sum_{m=1}^{\infty}(\frac{\rho^m}{R^{m+1}})\cos m(\theta-\phi), & R>\rho\\ T^e(R,\theta;\rho,\phi)=-\sum_{m=1}^{\infty}(\frac{R^{m-1}}{\rho^m})\cos m(\theta-\phi), & \rho>R\end{cases}$$

$$t_1=\begin{cases}\frac{1}{2\pi}\left[-\sum_{m=1}^{\infty}(\frac{\rho^{m-1}}{\bar{R}^m})\cos m(\bar{\theta}-\phi)\right], & R>\rho\\ \frac{1}{2\pi}\left[\frac{1}{\rho}+\sum_{m=1}^{\infty}(\frac{\bar{R}^m}{\rho^{m+1}})\cos m(\bar{\theta}-\phi)\right], & \rho>R\end{cases}$$

$$t_2(s)=-t_1(s)=\frac{1}{2\pi}\left[\sum_{m=1}^{\infty}(\frac{\rho^{m-1}}{R^m})\cos m(\bar{\theta}-\phi)\right]$$

known $\frac{\partial G(s,\xi)}{\partial n_x}\Rightarrow t=0$, unknown $G(s,\xi)=P_0+\sum_{n=1}^{\infty}P_n\cos n\phi+Q_n\sin n\phi$

$$0=\int_B T(s,x)G(s,\xi)dB(s)-\int_B U(s,x)t_2(x)dB(s)$$

$$0=\int_{-\pi}^{\pi}\left[\frac{1}{R}+\sum_{m=1}^{\infty}(\frac{\rho^{m}}{R^{m+1}})\cos m(\bar{\theta}-\phi)\right]\left[P_0+\sum_{n=1}^{\infty}P_n\cos n\bar{\theta}+Q_n\sin n\bar{\theta}\right]\bar{R}d\bar{\theta}$$

$$-\int_{B}\left[\ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\bar{R}}{\rho})^{m}\cos m(\bar{\theta}-\phi)\right]\left(\frac{1}{2\pi}\left[\sum_{m=1}^{\infty}(\frac{\rho^{m-1}}{\bar{R}^{m}})\cos m(\theta-\bar{\theta})\right]\right)\bar{R}d\bar{\theta}$$

$$0=-\sum_{m=1}^{\infty}(\frac{1}{a})\left(P_m\cos m\phi+Q_m\sin m\phi\right)+\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}(\frac{a^{m-1}}{\bar{R}^{m}})\left(\cos m\phi\cos m\theta+\sin m\phi\sin m\theta\right)$$

$$P_0=0\ ;P_m=-\frac{1}{2\pi m}(\frac{a}{\bar{R}})^{m}\cos m\theta\ ;Q_m=-\frac{1}{2\pi m}(\frac{a}{\bar{R}})^{m}\sin m\theta$$

$$2\pi u_2(x)=\int_{-\pi}^{\pi}\left[-\sum_{m=1}^{\infty}(\frac{a^{m-1}}{\rho^{m}})\cos m(\bar{\theta}-\phi)\right]\left[\sum_{m=1}^{\infty}\frac{1}{2\pi m}(\frac{a}{R})^{m}\cos m\theta\cos m\bar{\theta}+\frac{1}{2\pi m}(\frac{a}{R})^{m}\sin m\theta\sin m\bar{\theta}\right]ad\bar{\theta}$$

$$-\int_{B}\left[\left[\ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{a}{\rho})^{m}\cos m(\bar{\theta}-\phi)\right]\right]\left(\frac{1}{2\pi}\left[-\sum_{m=1}^{\infty}(\frac{a^{m-1}}{\bar{R}^{m}})\cos m(\theta-\bar{\theta})\right]\right)ad\bar{\theta}$$

$$u_2(x)=\frac{1}{2\pi}\left[-\left(\frac{1}{2m}(\frac{a^{m+1}}{\rho^{m}R^{m}})\cos m(\theta-\phi)\right)-\left(\frac{1}{2m}(\frac{a^{m+1}}{\rho^{m}R^{m}})\right)\cos m(\theta-\phi)\right]$$

$$u_2(x)=\frac{-1}{2\pi}\left(\frac{1}{m}(\frac{a^{m+1}}{\rho^{m}R^{m}})\cos m(\theta-\phi)\right)$$

$$u(x)=\begin{cases}\frac{1}{2\pi}\left[\ln\bar{R}-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{\bar{R}})^{m}\cos m(\bar{\theta}-\phi)\right]-\frac{1}{2\pi}\left(\frac{1}{m}(\frac{a^{m+1}}{\rho^{m}R^{m}})\cos m(\theta-\phi)\right), & \bar{R}\geq\rho\\ \frac{1}{2\pi}\left[\ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\bar{R}}{\rho})^{m}\cos m(\bar{\theta}-\phi)\right]-\frac{1}{2\pi}\left(\frac{1}{m}(\frac{a^{m+1}}{\rho^{m}R^{m}})\cos m(\theta-\phi)\right), & \bar{R}<\rho\end{cases}$$