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Rigid body mode and spurious mode in the dual boundary element formulation for the Laplace problems

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Abstract

In this paper, the general formulation for the static stiffness is analytically derived using the dual integral formulations. It is found that the same stiffness matrix is derived by using the integral equation no matter what the rigid body mode and the complementary solutions are superimposed in the fundamental solution. For the Laplace problem with a circular domain, the circulant was employed to derive the stiffness analytically in the discrete system. In deriving the static stiffness, the degenerate scale problem occurs when the singular influence matrix can not be inverted. The Fredholm alternative theorem and the SVD updating technique are employed to study the degenerate scale problem mathematically and numerically. The direct treatment in the matrix level is achieved to deal with the degenerate scale problems instead of using a modified fundamental solution. The addition of a rigid body term in the fundamental solution is found to shift the zero singular value for the singular matrix without disturbing the stiffness.

Keywords: Dual boundary integral equations; Rigid body mode; Laplace problem; Fredholm alternative theorem; SVD

1. Introduction

The concept of rigid body mode (RBM) has been used in the boundary element method (BEM) for determining the diagonal elements of the influence matrix [12]. For a half-plane problem, the displacements are referred to a fixed point where displacements are zero and this point is usually unknown. Later, Vable [23] described that the issue of RBM made the BEM less sensitive to errors or changes in the input data and arose as a consequence of implementing an algorithm. He also described the importance of RBM which affects the BEM analysis by using the direct and indirect methods [24]. The role of rigid body mode was also discussed in constructing the free–free flexibility matrix [16,17]. It is well-known that the Neumann problem leads to a singular matrices in boundary element implementation. The common remedy for such problems is to apply sufficient restraint on the body by prescribing displacements or temperatures at suitable points on its boundary [22]. How to remove the RBM from the discretized linear system has been published [1,22]. Also, Dumont [15] demonstrated the correct boundary element formulation in respect to the rigid body displacements that are presented in any fundamental solution. The nontrival solution for the singular matrix is found to be a rigid body term for the interior Neumann problem. This is both physically and mathematically realizable since rigid body modes are imbedded in the zero-eigenvalue matrix. However, in some special cases, the influence matrix of the weakly singular matrix (U) may be singular for the Dirichlet problem [9] when the geometry is special. The nonunique solution is physically unrealizable but stems from the zero eigenvalue of the influence matrix in the integral formulation mathematically [6].

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For the Dirichlet problem of potential problems [19], plate problems [10,11], and elasticity [4,19,20], the nonunique solution occurs in case of degenerate scale [5,19]. Also, the singular problem is embedded in the integral formulation for the problem with a degenerate scale. Recently, the RBM was added in the fundamental solution to overcome the degenerate scale problem in the BEM [6]. Also, a hypersingular equation can avoid the zero eigenvalues. Numerically speaking, the domain which results in a nonunique solution for the Dirichlet problem using the BEM is called a degenerate scale. Actually, the RBM removes the original degenerate scale to another one instead of completely eliminating it.

In the mathematical bibliography, the nonunique solution of BEM is related to Fredholm alternative theorem. Researchers have paid attention to the degenerate scale [19,20] or critical value [10,11] in the integral formulation and BEM algorithm. Zero eignvalue embedded in the influence matrix not only hinders the direct calculation of stiffness but also results in the rigid body mode. The singular matrix plays the negative role in determining the stiffness; however, the associated rigid body mode provides an alternative by adding it in the fundamental solution to avoid the zero eigenvalue. Felippa et al. [17] have constructed the free–free flexibility matrix as generalized stiffness inverse. A direct flexibility method developed in Felippa and Park [16] appears to be advantageous in the use of underintegrated elements without spurious-mode stabilization. Besides, the importance and use of RBM in BEM can improve the accuracy for the Neumann or traction problem when the equilibrium condition is not totally satisfied in the numerical manner [24]. The relation between global equilibrium and solvability of BEM was discussed by Blazquez et al. [1]. Although many related works on RBM and spurious mode have been done, the unification seems too loose. This is the main focus of this paper.

In this paper, the role of RBM and complementary solutions in deriving stiffness and degenerate scale will be examined. To overcome the degenerate scale problem, the relations between RBM and degenerate scale will be discussed. The stiffness matrices for a general structure will be derived by using the dual BEM. Three cases, rod, beam and membrane will be worked out for demonstration. The Fredholm alternative theorem and SVD updating technique will be employed to study the degenerate scale mathematically and numerically. Based on the unitary vectors in the SVD, a direct treatment in the influence matrix to deal with the degenerate scale problems in the linear algebra will be investigated. The relation between rigid body mode and spurious mode will be constructed in the linear algebra. Their roles will be examined. It is found that the addition of a rigid body mode can shift the zero singular value and make the stiffness easily determined.

2. Derivation of stiffness matrix for general structures using the dual BEM

Consider a homogeneous, isotropic, linear, elastic continuum with finite domain D bounded by boundary B, the governing equation for the displacement $u(x)$ can be written as

$$\mathscr{L}\{u(x)\} = 0, \quad x \in D, \tag{1}$$

where the operator $\mathscr{L}$ is

$$\mathscr{L}\{u\} = \begin{cases} \nabla^2 u, & \text{membrane}, \\ -\frac{\partial^2 u}{\partial x^2}, & \text{rod}, \\ -\frac{\partial^4 u}{\partial x^4}, & \text{beam}. \end{cases} \tag{2}$$

Based on the dual formulation, we can construct the boundary integral equations

$$\pi u(x) = \mathrm{CPV}\int_B T(s,x)u(s)\,\mathrm{d}B(s) - \mathrm{RPV}\int_B U(s,x)\frac{\partial u(s)}{\partial n_s}\,\mathrm{d}B(s), \quad x \in B \tag{3}$$

$$\pi t(x) = \mathrm{HPV}\int_B M(s,x)u(s)\,\mathrm{d}B(s) - \mathrm{CPV}\int_B L(s,x)\frac{\partial u(s)}{\partial n_s}\,\mathrm{d}B(s), \quad x \in B \tag{4}$$

where $t(x) = \partial u(x)/\partial n_x$, $U(s,x)$ is the fundamental solution, $T(s,x) = \partial U(s,x)/\partial n_s$, $L(s,x) = \partial U(s,x)/\partial n_x$ and $M(s,x) = \partial^2 U(s,x)/\partial n_s \partial n_x$, U, T, L and M are the four kernels as shown in Table 1 for U kernel, and RPV, CPV and HPV denote the Riemann principal value, Cauchy principal value and Hadamard or Mangler principal value, respectively. After discretizing Eqs. (3) and (4), we have

$$[U]\{t\} = [T]\{u\}, \tag{5}$$

$$[L]\{t\} = [M]\{u\}. \tag{6}$$

If $[U]$ and $[L]$ are invertible, the static stiffness can be obtained by

$$\{t\} = [K]_{UT}\{u\}, \tag{7}$$

$$\{t\} = [K]_{LM}\{u\}, \tag{8}$$

where $[K]_{UT}$ and $[K]_{LM}$ are the stiffness matrices obtained by the UT and LM equations, respectively, and

Table 1
The U kernels

	Rod	Beam	Membrane
$U(s,x)$	$\frac{1}{2}r$	$\frac{1}{12}r^3$	$\ln(r)$

$$[K]_{UT} = [U]^{-1}[T], \tag{9}$$

$$[K]_{LM} = [L]^{-1}[M]. \tag{10}$$

It is well known that the conventional BEM encounter the degenerate scale when the $[U]$ matrix is singular. In such case, the direct calculation for the stiffness matrix is not straightforward. In the following two sections, we will discuss the singular case in a special case of 2-D circular membrane.

3. Derivation of stiffness matrix for a circular membrane—special case

The governing equation for the two-dimensional Laplace equation is:

$$\nabla^2 u(x_1, x_2) = 0, \quad (x_1, x_2) \in D, \tag{11}$$

where ∇^2 is the Laplacian operator, and D is the domain. Based on the dual formulation [2], the unified null-field integral formulation for the Laplace equation using the direct method can be written as

$$0 = \int_B T(s,x)u(s)\,\mathrm{d}B(s) - \int_B U(s,x)t(s)\,\mathrm{d}B(s), \tag{12}$$

$$0 = \int_B M(s,x)u(s)\,\mathrm{d}B(s) - \int_B L(s,x)t(s)\,\mathrm{d}B(s), \tag{13}$$

where $t(s) = \partial u(s)/\partial n_s$ and B denotes the boundary enclosing D. For the exterior problem, we have $U(s,x) = U^{\mathrm{i}}(s,x)$, $T(s,x) = T^{\mathrm{i}}(s,x)$, $L(s,x) = L^{\mathrm{i}}(s,x)$ and $M(s,x) = M^{\mathrm{i}}(s,x)$. In case of interior problem, we have $U(s,x) = U^{\mathrm{e}}(s,x)$, $T(s,x) = T^{\mathrm{e}}(s,x)$, $L(s,x) = L^{\mathrm{e}}(s,x)$ and $M(s,x) = M^{\mathrm{e}}(s,x)$. The selected kernels are designed to have the null-field equation without the jump terms. The eight kernels of U^{i}, U^{e}, T^{i}, T^{e}, L^{i}, L^{e}, M^{i} and M^{e} can be obtained by using the degenerate kernels which will be elaborated on later. If the rigid body mode (c), and the linear terms ($pR\cos(\theta)$ and $qR\sin(\theta)$), are superimposed in the fundamental solution, we have $U_c(s,x) = U(s,x) + pR\cos(\theta) + qR\sin(\theta) + c$, where p, q and c are arbitrary constants. Based on the separable properties for the kernels, the kernel functions in the dual BEM can be expanded into degenerate forms as follows [3,21]:

$$U_c(s,x) = \begin{cases} U_c^{\mathrm{i}}(R,\theta;\rho,\phi) = \ln R + pR\cos(\theta) + qR\sin(\theta) + c \\ \qquad - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho}{R}\right)^m \cos(m(\theta-\phi)), & R > \rho, \\ U_c^{\mathrm{e}}(R,\theta;\rho,\phi) = \ln \rho + pR\cos(\theta) + qR\sin(\theta) + c \\ \qquad - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{R}{\rho}\right)^m \cos(m(\theta-\phi)), & R < \rho, \end{cases} \tag{14}$$

$$T_c(s,x) = \begin{cases} T_c^{\mathrm{i}}(R,\theta;\rho,\phi) = \frac{1}{R} + p\cos(\theta) + q\sin(\theta) \\ \qquad + \sum_{m=1}^{\infty} \frac{\rho^m}{R^{m+1}} \cos(m(\theta-\phi)), & R > \rho, \\ T_c^{\mathrm{e}}(R,\theta;\rho,\phi) = p\cos(\theta) + q\sin(\theta) \\ \qquad - \sum_{m=1}^{\infty} \frac{R^{m-1}}{\rho^m} \cos(m(\theta-\phi)), & R < \rho, \end{cases} \tag{15}$$

$$L_c(s,x) = \begin{cases} L_c^{\mathrm{i}}(R,\theta;\rho,\phi) = - \sum_{m=1}^{\infty} \frac{\rho^{m-1}}{R^m} \cos(m(\theta-\phi)), & R > \rho, \\ L_c^{\mathrm{e}}(R,\theta;\rho,\phi) = \frac{1}{\rho} + \sum_{m=1}^{\infty} \frac{R^m}{\rho^{m+1}} \cos(m(\theta-\phi)), & R < \rho, \end{cases} \tag{16}$$

$$M_c(s,x) = \begin{cases} M_c^{\mathrm{i}}(R,\theta;\rho,\phi) = \sum_{m=1}^{\infty} \frac{m\rho^{m-1}}{R^{m+1}} \cos(m(\theta-\phi)), & R > \rho, \\ M_c^{\mathrm{e}}(R,\theta;\rho,\phi) = \sum_{m=1}^{\infty} \frac{mR^{m-1}}{\rho^{m+1}} \cos(m(\theta-\phi)), & R < \rho, \end{cases} \tag{17}$$

where "i" and "e" denotes the interior point ($R > \rho$) and the exterior point ($R < \rho$), respectively, $x = (\rho, \phi)$ and $s = (R, \theta)$ in the polar coordinate as shown in Fig. 1. For a problem with a circular boundary, the boundary can be discretized into $2N$ constant elements with equal length. The linear algebraic dual equations can be obtained as shown below:

$$[U_c]_{2N\times 2N}\{t\}_{2N\times 1} = [T_c]_{2N\times 2N}\{u\}_{2N\times 1}, \tag{18}$$

$$[L_c]_{2N\times 2N}\{t\}_{2N\times 1} = [M_c]_{2N\times 2N}\{u\}_{2N\times 1}, \tag{19}$$

where $[U_c]$, $[T_c]$, $[L_c]$ and $[M_c]$ are the four influence matrices, $\{u\}$ and $\{t\}$ are the boundary data for the primary and the secondary fields, respectively. Based on the circular symmetry, the influence matrices for the discrete system are found to be circulants with the following forms [3,13,18,21],

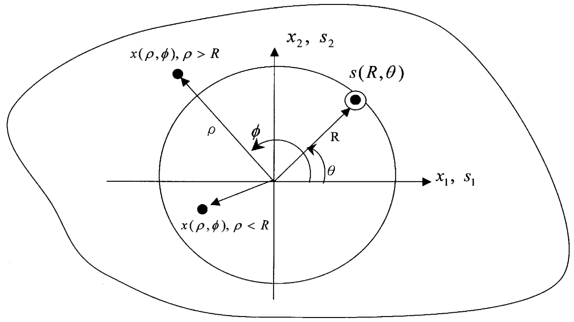

Fig. 1. Symbols in the degenerate kernels for the two-dimensional Laplace equation.

$$[U_c] = \begin{bmatrix} u_0 & u_1 & u_2 & \cdots & u_{2N-1} \\ u_{2N-1} & u_0 & u_1 & \cdots & u_{2N-2} \\ u_{2N-2} & u_{2N-1} & u_0 & \cdots & u_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_1 & u_2 & u_3 & \cdots & u_0 \end{bmatrix}, \tag{20}$$

$$[T_c] = \begin{bmatrix} t_0 & t_1 & t_2 & \cdots & t_{2N-1} \\ t_{2N-1} & t_0 & t_1 & \cdots & t_{2N-2} \\ t_{2N-2} & t_{2N-1} & t_0 & \cdots & t_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & t_3 & \cdots & t_0 \end{bmatrix}, \tag{21}$$

$$[L_c] = \begin{bmatrix} l_0 & l_1 & l_2 & \cdots & l_{2N-1} \\ l_{2N-1} & l_0 & l_1 & \cdots & l_{2N-2} \\ l_{2N-2} & l_{2N-1} & l_0 & \cdots & l_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_1 & l_2 & l_3 & \cdots & l_0 \end{bmatrix}, \tag{22}$$

$$[M_c] = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{2N-1} \\ m_{2N-1} & m_0 & m_1 & \cdots & m_{2N-2} \\ m_{2N-2} & m_{2N-1} & m_0 & \cdots & m_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & m_3 & \cdots & m_0 \end{bmatrix}, \tag{23}$$

where u_m, t_m, l_m, and m_m are shown below,

$$\begin{aligned} u_m &= \int_{\left(m-\frac{1}{2}\right)\Delta\theta}^{\left(m+\frac{1}{2}\right)\Delta\theta} U_c^{\mathrm{e}}(R,\theta;\rho,0)\rho\,\mathrm{d}\theta \\ &\approx U_c^{\mathrm{e}}(R,\theta_m;\rho,0)\rho\Delta\theta, \quad m = 0,1,2,\ldots,2N-1, \end{aligned} \tag{24}$$

$$\begin{aligned} t_m &= \int_{\left(m-\frac{1}{2}\right)\Delta\theta}^{\left(m+\frac{1}{2}\right)\Delta\theta} T_c^{\mathrm{e}}(R,\theta;\rho,0)\rho\,\mathrm{d}\theta \\ &\approx T_c^{\mathrm{e}}(R,\theta_m;\rho,0)\rho\Delta\theta, \quad m = 0,1,2,\ldots,2N-1, \end{aligned} \tag{25}$$

$$\begin{aligned} l_m &= \int_{\left(m-\frac{1}{2}\right)\Delta\theta}^{\left(m+\frac{1}{2}\right)\Delta\theta} L_c^{\mathrm{e}}(R,\theta;\rho,0)\rho\,\mathrm{d}\theta \\ &\approx L_c^{\mathrm{e}}(R,\theta_m;\rho,0)\rho\Delta\theta, \quad m = 0,1,2,\ldots,2N-1, \end{aligned} \tag{26}$$

$$\begin{aligned} m_m &= \int_{\left(m-\frac{1}{2}\right)\Delta\theta}^{\left(m+\frac{1}{2}\right)\Delta\theta} M_c^{\mathrm{e}}(R,\theta;\rho,0)\rho\,\mathrm{d}\theta \\ &\approx M_c^{\mathrm{e}}(R,\theta_m;\rho,0)\rho\Delta\theta, \quad m = 0,1,2,\ldots,2N-1, \end{aligned} \tag{27}$$

in which $\Delta\theta = 2\pi/2N$ and $\theta_m = m\Delta\theta$ and setting $\phi = 0$ without loss of generality. The four matrices in Eqs. (20)–(23) have only $N+1$ different elements since rotation symmetry with circulant property is reserved. By introducing the following bases for circulants, $I, (C_{2N})^1, (C_{2N})^2, \ldots, (C_{2N})^{2N-1}$, we can expand matrix $[U_c]$ into

$$[U_c] = u_0 I + u_1 (C_{2N})^1 + u_2 (C_{2N})^2 + \cdots + u_{2N-1}(C_{2N})^{2N-1}, \tag{28}$$

where

$$C_{2N} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{2N\times 2N}. \tag{29}$$

Based on the similar properties for the matrices of $[U_c]$ and $[C_{2N}]$, we have

$$\begin{aligned} &\lambda_l^{[U_c]} = u_0 + u_1\alpha_l + u_2\alpha_l^2 + \cdots + u_{2N-1}\alpha_l^{2N-1}, \\ &\quad l = 0, \pm 1, \pm 2, \ldots, \pm N-1, \pm N, \end{aligned} \tag{30}$$

where $\lambda_l^{[U_c]}$ and α_l are the eigenvalues for $[U_c]$ and $[C_{2N}]$. It is easily found that the eigenvalues for the circulants $[C_{2N}]$, are the roots for $\alpha^{2N} = 1$ as shown below:

$$\begin{aligned} &\alpha_n = \mathrm{e}^{\mathrm{i}(2\pi n)/(2N)}, \quad n = 0, \pm 1, \pm 2, \ldots, \pm N-1, \pm N, \\ &\quad \text{or} \quad n = 0, 1, 2, \ldots, 2N-1. \end{aligned} \tag{31}$$

The eigenvectors are

$$\{v_l\} = \begin{Bmatrix} 1 \\ \alpha_l \\ \alpha_l^2 \\ \vdots \\ \alpha_l^{2N-1} \end{Bmatrix}. \tag{32}$$

The transformation matrix $[\Phi]$ is composed of the eigenvectors as

$$\begin{aligned} [\Phi] &= \frac{1}{\sqrt{2N}} \begin{bmatrix} 1 & v_l & v_l^2 & \cdots & v_l^{2N-1} \end{bmatrix} \\ &= \frac{1}{\sqrt{2N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{2N-1} \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_{2N-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_0^{2N-1} & \alpha_1^{2N-1} & \alpha_2^{2N-1} & \cdots & \alpha_{2N-1}^{2N-1} \end{bmatrix}_{2N\times 2N}. \end{aligned} \tag{33}$$

Substituting Eq. (31) into Eq. (30), we have

$$\begin{aligned} \lambda_l^{[U_c]} &= \sum_{m=0}^{2N-1} u_m (\alpha_l)^m = \sum_{m=0}^{2N-1} u_m \mathrm{e}^{\mathrm{i}ml(2\pi)/(2N)} \\ &= \sum_{m=0}^{2N-1} U_c^{\mathrm{e}}(R,\theta;R,0) R\Delta\theta \mathrm{e}^{\mathrm{i}ml\Delta\theta} \\ &= R\sum_{m=0}^{2N-1} U_c^{\mathrm{e}}(R,\theta;R,0)\mathrm{e}^{\mathrm{i}ml\Delta\theta}\Delta\theta. \end{aligned} \tag{34}$$

When N approaches infinity, the Riemann sum in Eq. (34) can be transformed to the following integral,

$$\lambda_l^{[U_c]} = R\int_0^{2\pi} U_c^{\mathrm{e}}(R,\theta;R,0)\mathrm{e}^{\mathrm{i}l\theta}\,\mathrm{d}\theta. \quad (35)$$

By substituting U_c^{e} kernel of Eq. (14) into Eq. (35), we have

$$\begin{aligned}\lambda_l^{[U_c]} &= R\int_0^{2\pi}\left((\ln R + pR\cos(\theta) + qR\sin(\theta) + c) - \sum_{m=1}^{\infty}\frac{1}{m}\cos(m\theta)\right)\mathrm{e}^{\mathrm{i}l\theta}\,\mathrm{d}\theta \\ &= \begin{cases} 2\pi R(\ln R + c), & l = 0, \\ -\pi\frac{R}{|l|}, & l = \pm 1, \pm 2, \ldots, \pm(N-1), N. \end{cases}\end{aligned} \quad (36)$$

Also, we have

$$\begin{aligned}\lambda_l^{[T_c]} &= \int_0^{2\pi}\left((p\cos(\theta) + q\sin(\theta)) - \sum_{m=1}^{\infty}\frac{1}{R}\cos(m\theta)\right)\mathrm{e}^{\mathrm{i}l\theta}\,\mathrm{d}\theta \\ &= \begin{cases} 0, & l = 0, \\ -\pi, & l = \pm 1, \pm 2, \ldots, \pm(N-1), N. \end{cases}\end{aligned} \quad (37)$$

Similarly, we can obtain the eigenvalues for the other influence matrices,

$$\lambda_l^{[L_c]} = \begin{cases} 2\pi, & l = 0, \\ \pi, & l = \pm 1, \pm 2, \ldots, \pm(N-1), N, \end{cases} \quad (38)$$

$$\lambda_l^{[M_c]} = \begin{cases} 0, & l = 0, \\ \pi\frac{|l|}{R}, & l = \pm 1, \pm 2, \ldots, \pm(N-1), N. \end{cases} \quad (39)$$

By employing the SVD technique for a circular case with $\Phi = \Psi$, we can transform the dual equations in Eq. (18) and Eq. (19) into,

$$\Phi\Sigma\Phi^{\mathrm{T}}\{t\} = \Phi\Sigma\Phi^{\mathrm{T}}\{u\}, \quad (40)$$

$$\Phi\Sigma\Phi^{\mathrm{T}}\{t\} = \Phi\Sigma\Phi^{\mathrm{T}}\{u\}, \quad (41)$$

where Φ and Ψ are the left and right unitary vector, respectively, the superscript "T" denotes the transpose and

$$\Sigma_{U_c} = \begin{bmatrix} 2\pi R(\ln R + c) & 0 & \cdots & \cdots & 0 \\ 0 & -\pi R & 0 & \cdots & \cdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & -\pi\left(\frac{R}{N-1}\right) & 0 \\ 0 & 0 & \cdots & 0 & -\pi\left(\frac{R}{N}\right) \end{bmatrix}_{2N\times 2N}, \quad (42)$$

$$\Sigma_{T_c} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & -\pi & 0 & \cdots & \cdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \cdots & 0 & -\pi & 0 \\ 0 & 0 & \cdots & 0 & -\pi \end{bmatrix}_{2N\times 2N}, \quad (43)$$

$$\Sigma_{L_c} = \begin{bmatrix} 2\pi & 0 & \cdots & \cdots & 0 \\ 0 & \pi & 0 & \cdots & \cdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \pi & 0 \\ 0 & 0 & \cdots & 0 & \pi \end{bmatrix}_{2N\times 2N}, \quad (44)$$

$$\Sigma_{M_c} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & \frac{\pi}{R} & 0 & \cdots & \cdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \pi\left(\frac{N-1}{R}\right) & 0 \\ 0 & 0 & \cdots & 0 & \pi\left(\frac{N}{R}\right) \end{bmatrix}_{2N\times 2N}. \quad (45)$$

The static stiffness matrix can be expressed as

$$[K] = [U_c]^{-1}[T_c] = [L_c]^{-1}[M_c], \quad (46)$$

where $\{t\} = [K]\{u\}$ in Eqs. (40) and (41) can be rewritten as,

$$\{t\} = \Phi\Sigma_{U_c}^{-1}\Sigma_{T_c}\Phi^{\mathrm{T}}\{u\}, \quad (47)$$

$$\{t\} = \Phi\Sigma_{L_c}^{-1}\Sigma_{M_c}\Phi^{\mathrm{T}}\{u\}. \quad (48)$$

The diagonal matrix, Σ_K, can be determined by

$$\Sigma_K = \Sigma_{U_c}^{-1}\Sigma_{T_c} = \Sigma_{L_c}^{-1}\Sigma_{M_c}. \quad (49)$$

Based on the inverse for the matrices of Σ_{U_c} and Σ_{L_c}, we have

$$\Sigma_{U_c}^{-1} = \begin{bmatrix} \frac{1}{2\pi R(\ln R + c)} & 0 & \cdots & \cdots & 0 \\ 0 & -\frac{1}{\pi R} & 0 & \cdots & \cdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & -\left(\frac{N-1}{\pi R}\right) & 0 \\ 0 & 0 & \cdots & 0 & -\left(\frac{N}{\pi R}\right) \end{bmatrix}_{2N\times 2N}, \quad (50)$$

$$\Sigma_{L_c}^{-1} = \begin{bmatrix} \frac{1}{2\pi} & 0 & \cdots & \cdots & 0 \\ 0 & \frac{1}{\pi} & 0 & \cdots & \cdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \frac{1}{\pi} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{\pi} \end{bmatrix}_{2N\times 2N}. \quad (51)$$

It is interesting to find that the static stiffness matrix for the circular case is shown below,

$$[K] = \Phi\Sigma_K\Phi^{\mathrm{T}}, \quad (52)$$

where

$$\Sigma_K = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & \frac{1}{R} & 0 & \cdots & \cdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & \frac{N-1}{R} & 0 \\ 0 & 0 & \cdots & 0 & \frac{N}{R} \end{bmatrix}_{2N\times 2N}. \quad (53)$$

Table 2
The stiffness matrices and degenerate scales for a rod, a beam and a circular membrane

	The fundamental solution	Stiffness matrix	Degenerate scale
1-D rod	$U_c(s,x) = \frac{1}{2}\lvert x-s\rvert + as + c$	$[K] = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	$(1+2a)l = -4c$
1-D beam	$U_c(s,x) = \frac{1}{12}\lvert x-s\rvert^3 + as + bs^2 + ds^3 + c$	$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$	$(1+12d)l^3 - 24al = 48c$
2-D circular membrane	$U_c(s,x) = \ln R + pR\cos(\theta) + qR\sin(\theta) + c$	$[K] = \Phi \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & \frac{1}{R} & 0 & \cdots & \cdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & 0 & \frac{N-1}{R} & 0 \\ 0 & 0 & \cdots & 0 & \frac{N}{R} \end{bmatrix}_{2N\times 2N} \Phi^{\mathrm{T}}$	$\ln R = -c$

We obtain the stiffness matrix analytically no matter that the values of p, q and c are specified. However, the $[U_c]$ matrix may be singular once $\ln R + c = 0$ in Eq. (50). In this case, the stiffness matrix can only be obtained analytically. If the numerical instability can be suppressed, we can determine the stiffness numerically. This indicates that the rigid body mode and the complementary solutions result in a degenerate scale such that $\ln R = -c$. In the conventional BEM ($c = 0$), the radius of a circular membrane equal to 1 is a degenerate scale problem. To deal with this problem, the truncated singular value decomposition (TSVD) technique in conjunction with the concept of pseudo-inverse [6,8] have been employed to calculate the stiffness matrix [7]. It is found that the same stiffness matrix can be obtained from the auxiliary system by superimposing $pR\cos(\theta) + qR\sin(\theta) + c$ in the dual integral formulation. All the above results are summarized in Table 2. Instead of using *UT* formulation, it is found that *LM* formulation can determine the stiffness for any case as shown in Eq. (51). Another alternative to deal with the degenerate scale problem is CHEEF method [5].

4. Discussions of the rigid body mode and spurious mode in case of degenerate scale using the Fredholm alternative theorem and SVD updating technique

In the above analysis, we find that the degenerate scale stems from a singular influence matrix. The relations between the rigid body mode and the degenerate scale will be studied mathematically and numerically in this section.

Fredholm alternative theorem:

The equation $[H]\{u\} = \{f\}$ has a unique solution if and only if the only continuous solution to the homogeneous equation

$$[H]\{u\} = \{0\} \tag{54}$$

is $\{u\} = \{0\}$. Alternatively, the homogeneous equation has at least one solution if the homogeneous adjoint equation

$$[H]^{\dagger}\{\phi\} = \{0\} \tag{55}$$

has a nontrivial solution $\{\phi\}$, where $[H]^{\dagger}$ is the transpose conjugate matrix of $[H]$ and $\{\phi\}$ must satisfy the constraint $(\{f\}^{\dagger}\{\phi\} = 0)$. If the matrix $[H]$ is real, the transpose conjugate of a matrix is equal to transpose only [14], i.e., $[H]^{\dagger} = [H]^{\mathrm{T}}$. By using the UT formulation, we have

$$UT\ formulation: \quad [U]\{t\} = [T]\{u\} = \{f\}. \tag{56}$$

According to the Fredholm alternative theorem, Eq. (56) has at least one solution for $\{t\}$ if the homogeneous adjoint equation

$$[U]^{\mathrm{T}}\{\phi_1\} = \{0\} \tag{57}$$

has a nontrivial solution $\{\phi_1\}$, in which the constraint $(f^{\mathrm{T}}\phi_1 = 0)$ must be satisfied. By substituting Eq. (54) to $\{f\}^{\mathrm{T}}\{\phi_1\} = 0$, we obtain

$$\{u\}^{\mathrm{T}}[T]^{\mathrm{T}}\{\phi_1\} = 0. \tag{58}$$

Since $\{u\}$ is an arbitrary vector for the Dirichlet problem, we have

$$[T]^{\mathrm{T}}\{\phi_1\} = \{0\}. \tag{59}$$

Based on Eqs. (57) and (59), we have

$$\begin{bmatrix} [U]^{\mathrm{T}} \\ [T]^{\mathrm{T}} \end{bmatrix}\{\phi_1\} = \{0\} \quad \text{or} \quad \{\phi_1\}^{\mathrm{T}}[[U] \quad [T]] = \{0\}. \tag{60}$$

Eq. (60) indicates that the two matrices have the same spurious mode $\{\phi_1\}$ corresponding to the same zero singular value when a degenerate scale problem occurs.

By using the SVD technique for the $[U]^{\mathrm{T}}$ and $[T]^{\mathrm{T}}$ matrices, we have

$$\begin{aligned}[U]^{\mathrm{T}} &= \Psi_U \Sigma_U \Phi_U^{\mathrm{T}},\\ [T]^{\mathrm{T}} &= \Psi_T \Sigma_T \Phi_T^{\mathrm{T}}.\end{aligned} \tag{61}$$

By substituting Eq. (61) into Eq. (60), we have

$$\begin{aligned}&\sum_j \sigma_j^{(U)}\{\psi_j^{(U)}\}\{\phi_j^{(U)}\}^{\mathrm{T}}\{\phi_i^{(U)}\}\\ &\quad = \{0\}\overrightarrow{\phi_i \cdot \phi_j = \delta_{ij}}\sigma_i^{(U)}\{\psi_i^{(U)}\} = \{0\}, \quad (i \text{ no sum}),\\ &\sum_j \sigma_j^{(T)}\{\psi_j^{(T)}\}\{\phi_j^{(T)}\}^{\mathrm{T}}\{\phi_i^{(T)}\}\\ &\quad = \{0\}\overrightarrow{\phi_i \cdot \phi_j = \delta_{ij}}\sigma_i^{(T)}\{\psi_i^{(T)}\} = \{0\}, \quad (i \text{ no sum}),\end{aligned} \tag{62}$$

where the superscripts (T) and (U) denotes the $[T]$ and $[U]$ matrix, respectively, $\{\phi_i\}$ and $\{\psi_i\}$ are the orthonormal bases, $\sigma_i^{(U)}$ and $\sigma_i^{(T)}$ are the zero singular values of $[U]$ and $[T]$ matrices, respectively. We can easily extract the eigensolutions since there exists the same spurious mode $\{\phi_i\}$ corresponding to the zero singular values ($\sigma_i^{(U)} = \sigma_i^{(T)} = 0$).

Rod and beam cases:

The $[U]$ matrix for a rod is singular when $(1+2a)l = -4c$. This results in the degenerate scale problem. Without loss of generality of $l = 1$, we can reconstruct the *UT* formulation as shown below,

$$\begin{aligned}&\begin{bmatrix} \frac{1}{2}+a & -\frac{1}{2}-a \\ -\frac{1}{2}+a & \frac{1}{2}-a \end{bmatrix}\begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix}\\ &\quad = \begin{bmatrix} c & -\frac{1}{2}-a-c \\ \frac{1}{2}+c & -a-c \end{bmatrix}\begin{Bmatrix} t(0) \\ t(1) \end{Bmatrix}.\end{aligned} \tag{63}$$

According to the Fredholm alternative theorem, the degenerate scale depends on the rigid body term. A special case of $a = 0$ and $c = -\frac{1}{4}$ results in a degenerate scale. By employing the SVD technique, we have

$$\begin{aligned}[T] &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\\ &= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \mathbf{0} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{\mathrm{T}},\end{aligned} \tag{64}$$

$$\begin{aligned}[U] &= \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}\\ &= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \mathbf{0} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{\mathrm{T}}.\end{aligned} \tag{65}$$

The spurious mode $\{\phi_1\}$to satisfy Eq. (60) for $[U]$ and $[T]$ in Eqs. (64) and (65), respectively, is found to be

$$\{\phi_1\} = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}_{2\times 1}, \tag{66}$$

where

$$U^{\mathrm{T}}\{\phi_1^{(U)}\} = 0\{\psi_1^{(U)}\}, \quad U\{\psi_1^{(U)}\} = 0\{\phi_1^{(U)}\}. \tag{67}$$

It is interesting to find that the rigid body mode is shown below,

$$\{\psi_1^{(T)}\} = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}_{2\times 1}, \tag{68}$$

where

$$[T]\{\psi_1^{(T)}\} = \{0\}. \tag{69}$$

The spurious mode $\{\psi_1^{(U)}\}$ and the rigid body mode $\{\psi_1^{(T)}\}$ are shown in Fig. 2.

Similarly, the spurious mode $\{\phi_1^{(U)}\}$ and the two rigid body modes $\{\psi_1^{(T)}\}$ and $\{\psi_2^{(T)}\}$ are determined and are shown in Table 3 and Fig. 2 for the beam case.

Circular membrane case:

The $[U]$ matrix for the 2-*D* circular membrane is singular when $\ln R + c = 0$. This results in the degenerate scale problem. Using the degenerate kernels and circulants for a circular membrane, we can reconstruct the *UT* formulation as shown below,

$$\Phi_{U_c}\Sigma_{U_c}\Phi_{U_c}^{\mathrm{T}}\{t\} = \Phi_{T_c}\Sigma_{T_c}\Phi_{T_c}^{\mathrm{T}}\{u\}, \tag{70}$$

where Σ_{U_c} and Σ_{T_c} can be found in Eqs. (42) and (43). According to the Fredholm alternative theorem in conjunction with SVD updating technique, we have the spurious mode $\{\phi_1\}$

$$\{\phi_1\} = \frac{1}{\sqrt{2N}}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix}_{2N\times 1}, \tag{71}$$

where $\{\phi_1\}$ satisfies

$$\begin{bmatrix} [U]^{\mathrm{T}} \\ [T]^{\mathrm{T}} \end{bmatrix}\{\phi_1\} = \{0\}. \tag{72}$$

After we obtain the spurious mode $\{\phi_1\}$, we can extract $\{\psi_1^{(U)}\}$ by

$$[U]\{\psi_1^{(U)}\} = \{0\}. \tag{73}$$

It is interesting to find that the rigid body mode is shown below,

$$\{\psi_1^{(U)}\} = \frac{1}{\sqrt{2N}}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix}_{2N\times 1}. \tag{74}$$

Although we have solved the singular problem by superimposing a rigid body mode (c) in the fundamental solution, this can be done in the matrix level by using

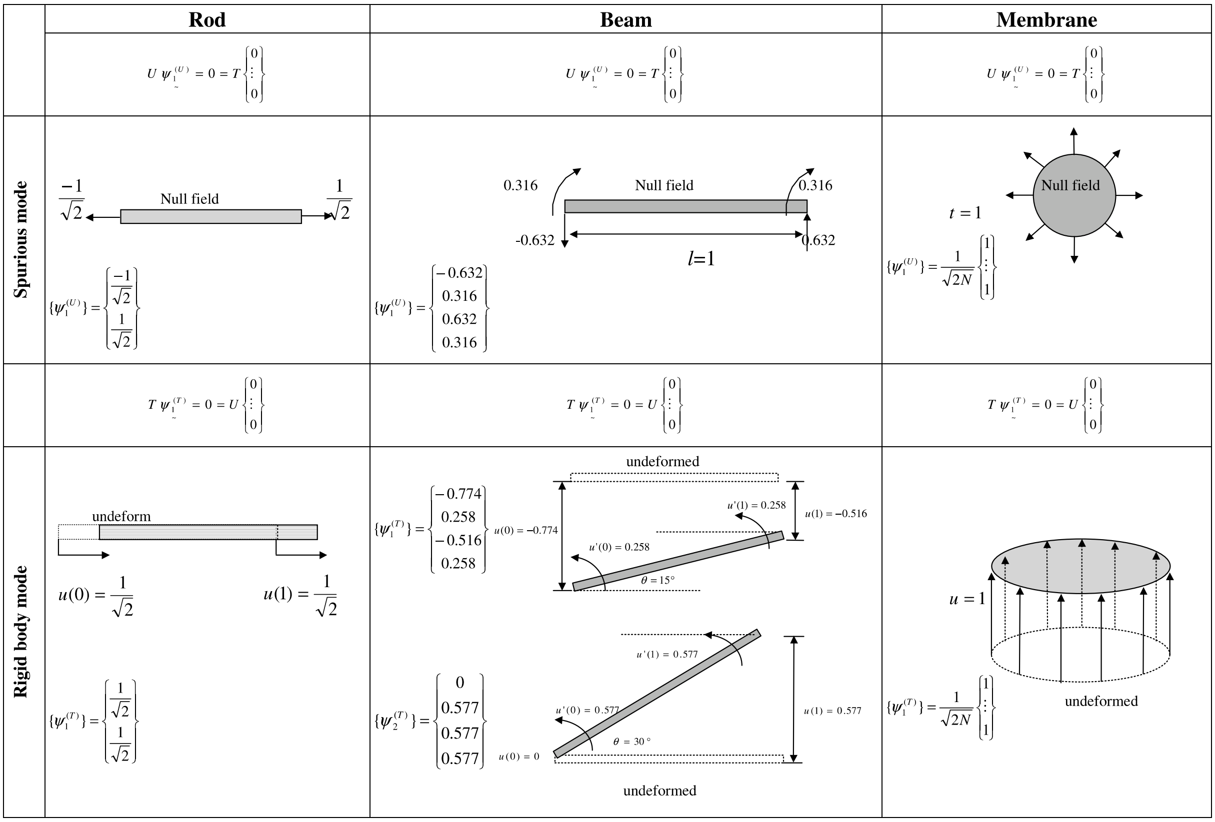

Fig. 2. The rigid body mode and spurious mode for a beam and a rod in BEM.

Table 3
The rigid body mode and spurious mode for a rod, a beam and a circular membrane in case of degenerate scale

	Degenerate scale	Spurious mode (Dirichlet problem)	Special case	Rigid body mode (Neumann problem)
1-D rod	$(1+2a)l = -4c$	$\begin{bmatrix}U^{\mathrm{T}}\\T^{\mathrm{T}}\end{bmatrix}_{4\times2}\{\phi_1\}_{2\times1} = 0$	$a=0,\ c=-\frac{1}{4},\ l=1$ $\{\phi_1\}=\{\psi_1\}=\frac{1}{\sqrt{2}}\begin{Bmatrix}1\\1\end{Bmatrix}_{2\times1}$	$\begin{bmatrix}T^{\mathrm{T}}\\M^{\mathrm{T}}\end{bmatrix}_{4\times2}\times\{\psi_1\}_{2\times1}=\{0\}$
1-D beam	$(1+12d)l^3 - 24al = 48c$		$a=0,\ b=0,\ c=\frac{1}{48},\ d=0,\ l=1$	
		$\begin{bmatrix}U^{\mathrm{T}}\\T^{\mathrm{T}}\end{bmatrix}_{8\times4}\{\phi_1\}_{4\times1} = 0$	$\Leftarrow \{\phi_1\}=\begin{Bmatrix}-0.632\\-0.632\\-0.316\\0.316\end{Bmatrix}_{4\times1}$ $\{\psi_1\}=\begin{Bmatrix}-0.774\\0.258\\-0.516\\0.258\end{Bmatrix}_{4\times1} \Rightarrow$	$\begin{bmatrix}T^{\mathrm{T}}\\M^{\mathrm{T}}\end{bmatrix}_{8\times4}\times\{\psi_1\}_{4\times1}=\{0\}$
			$\{\psi_2\}=\begin{Bmatrix}0\\0.577\\0.577\\0.577\end{Bmatrix}_{4\times1} \Rightarrow$	$\begin{bmatrix}T^{\mathrm{T}}\\M^{\mathrm{T}}\end{bmatrix}_{8\times4}\times\{\psi_2\}_{4\times1}=\{0\}$
2-D circular membrane	$\ln R = -c$	$\begin{bmatrix}U^{\mathrm{T}}\\T^{\mathrm{T}}\end{bmatrix}_{4N\times2N}\{\phi_1\}_{2N\times1} = 0$	$\{\phi_1\}=\{\psi_1\}=\frac{1}{\sqrt{2N}}\begin{Bmatrix}1\\1\\1\\\vdots\\1\end{Bmatrix}_{2N\times1}$	$\begin{bmatrix}T^{\mathrm{T}}\\M^{\mathrm{T}}\end{bmatrix}_{4N\times2N}\times\{\psi_1\}_{2N\times1}=\{0\}$

$$\left[U + \alpha\{\phi_1^{(U)}\}\{\psi_1^{(U)}\}^{\mathrm{T}}\right]\{t\} = [T]\{u\}, \tag{75}$$

where $\alpha = 2Ncl_i$, l_i is the element length. The α is a nonzero constant which can be understood as the rigid body motion. Eq. (75) can be rewritten as

$$\left[\sum \sigma_i^{(U)}\{\phi_i^{(U)}\}\{\psi_i^{(U)}\}^{\mathrm{T}} + \alpha\{\phi_1^{(U)}\}\{\psi_1^{(U)}\}^{\mathrm{T}}\right]\{t\} = \left[\sum \sigma_i^{(T)}\{\phi_i^{(T)}\}\{\psi_i^{(T)}\}^{\mathrm{T}}\right]\{u\}. \tag{76}$$

To demonstrate that the solution is reserved in the modified form of Eq. (76) in comparison with Eq. (70), the proof is shown below.

Proof. To prove that the solution for Eqs. (70) and (75) are the same no matter what the value α is. By expressing the boundary data $\{u\}$ and $\{t\}$ in terms of generalized coordinates $\{p\}$ and $\{q\}$ for unitary vectors, we have

$$\{u\} = \Psi_T\{q\}, \tag{77}$$

$$\{t\} = \Psi_U\{p\}. \tag{78}$$

Substituting Eqs. (77) and (78) into Eqs. (70) and (75), we have

$$0p_1\{\phi_1\} + \sigma_2 p_2\{\phi_2^{(U)}\} + \cdots + \sigma_N p_N\{\phi_N^{(U)}\} = 0q_1\{\phi_1\} + \bar{\sigma}_2 q_2\{\phi_2^{(T)}\} + \cdots + \bar{\sigma}_N q_N\{\phi_N^{(T)}\}, \tag{79}$$

$$\alpha p_1\{\phi_1\} + \sigma_2 p_2\{\phi_2^{(U)}\} + \cdots + \sigma_N p_N\{\phi_N^{(U)}\} = 0q_1\{\phi_1\} + \bar{\sigma}_2 q_2\{\phi_2^{(T)}\} + \cdots + \bar{\sigma}_N q_N\{\phi_N^{(T)}\}, \tag{80}$$

where σ_i and $\bar{\sigma}_i$ are the singular values of U and T matrices, $\{\phi_i^{(U)}\}$ and $\{\phi_i^{(T)}\}$ are the left unitary vectors in Φ_U and Φ_T as shown below:

$$[\Phi_U] = \left[\{\phi_1\} \quad \{\phi_2^{(U)}\} \quad \cdots \quad \{\phi_N^{(U)}\}\right], \tag{81}$$

$$[\Phi_T] = \left[\{\phi_1\} \quad \{\phi_2^{(T)}\} \quad \cdots \quad \{\phi_N^{(T)}\}\right]. \tag{82}$$

By taking the inner product for Eqs. (79) and (80) with respect to $\{\phi_1\}^{\mathrm{T}}$, we have

$$0 \cdot p_1 = 0 \cdot q_1, \tag{83}$$

$$\alpha p_1 = 0 \cdot q_1. \tag{84}$$

The degenerate scale results in a indefinite form Eq. (83) where p_1 can not be determined. After the shifting process in Eq. (80), p_1 is found to be zero in Eq. (84). By taking the inner product of Eqs. (79) and (80) with respect to $\{\phi_j^{(U)}\}$ $(j \geqslant 2)$, both equations reduce to the same one,

$$p_j = \sum_{k=2}^{N} \bar{\sigma}_k q_k C_{jk}, \tag{85}$$

where

$$C_{jk} = \phi_j^{(U)} \cdot \phi_k^{(T)}. \tag{86}$$

This indicates that the shifting process in Eq. (70) does not perturb the solution for the original Eq. (75). $\square$

5. Conclusions

Based on the dual BEM, we have derived analytically the same stiffness no matter that the rigid body mode and the complementary solutions are superimposed in the fundamental solution for general structures. It is found that degenerate scale depends on the fundamental solution which is superimposed by the rigid body mode and the complementary solution. Also, it is interesting to find that the linear term can not change the position of degenerate scale for a circular membrane. Based on the Fredholm theorem and the SVD updating technique, the spurious mode was derived for the degenerate scale problem. By shifting the zero singular value using the rigid body mode and spurious mode in the matrix level, the singular matrix can be desingularized which is found to be equivalent by adding a rigid body term in the fundamental solution. The relations between the rigid body mode and the degenerate scale were also examined. Illustrative numerical examples including rod, beam and membrane were demonstrated to show the validity of the formulation.

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