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Analytical study and numerical experiments for degenerate scale problems in the boundary element method for two-dimensional elasticity

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SUMMARY

For a plane elasticity problem, the boundary integral equation approach has been shown to yield a non-unique solution when geometry size is equal to a degenerate scale. In this paper, the degenerate scale problem in the boundary element method (BEM) is analytically studied using the method of stress function. For the elliptic domain problem, the numerical difficulty of the degenerate scale can be solved by using the hypersingular formulation instead of using the singular formulation in the dual BEM. A simple example is shown to demonstrate the failure using the singular integral equations of dual BEM. It is found that the degenerate scale also depends on the Poisson's ratio. By employing the hypersingular formulation in the dual BEM, no degenerate scale occurs since a zero eigenvalue is not imbedded in the influence matrix for any case. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: boundary elements; elasticity; degenerate scale; degenerate kernel; Airy stress function

1. INTRODUCTION

The boundary element method (BEM) has been applied to solve the potential problems, e.g. scalar potential for the Laplace equation and vector potential for the Navier equation in the recent decades [1]. It is well known that rigid body motion test or the so-called use of a simple solution can be employed to check the singular matrices of the strongly singular and hypersingular kernels for the problems without degenerate boundaries. In such a case, singular matrix occurs physically and mathematically. The non-trivial solution for the singular matrix can be physically realized to be a rigid body mode for the interior traction problem. However, for problems with special scale of geometry shape, the influence matrix of the weakly singular

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kernel may be singular for the displacement specified problem [2]. The non-unique solution is not physically realizable but stems from the zero eigenvalue imbedded in the influence matrix of the discrete system using the boundary element formulation. The special boundary geometry which results in a non-unique solution for plane elasticity problems is also called a degenerate scale in a manner similar to the scalar potential case in Reference [3]. For several problems with specific boundary conditions, some studies for plane elasticity problems [4–7] and potential problems [2, 5, 8] have been done. A rigorous study was proposed mathematically by Kuhn [9] and Constanda [10, 11] for the occurring mechanism of degenerate scale. Also, a simple treatment to sort out the problem was proposed by superimposing a rigid body term in the fundamental solution. Nevertheless, this approach results in a new degenerate scale instead of the original one. The difficulties due to the non-uniqueness of solutions were also overcome by the necessary and sufficient boundary integral formulation [6] and the boundary contour method [7]. However, the boundary conditions in their cases are either the Dirichlet or mixed type and must be constant along the boundary. Fictitious BEM also results in the degenerate scale problem when selecting a special fictitious boundary, and has been proven to yield non-unique solutions for two-dimensional potential problems [12]. Nevertheless, no general proof has been carried out for the degenerate scale problem.

In this paper, the degenerate scale problem for two-dimensional elasticity with an elliptic domain in BEM will be studied analytically and numerical experiments will be performed. Based on the method of stress function, the singularity pattern distributed along a boundary resulting in a null field will introduce the problem of degenerate scale. Also, the role of hypersingular formulation will be examined for solving the degenerate scale problems.

2. REVIEW OF DUAL BOUNDARY INTEGRAL EQUATIONS FOR ELASTICITY

Let (b_i, u_i, t_i) and (b_i^*, t_i^*, u_i^*) be two equilibrium states in a linearly elastic body where b_i and b_i^* are the body forces; t_i and t_i^* denote the boundary tractions; and u_i and u_i^* are the displacements. Betti's law gives [13]

$$\int_D (u_i b_i^* - u_i^* b_i)\, \mathrm{d}V = -\int_B (u_i t_i^* - u_i^* t_i)\, \mathrm{d}B \tag{1}$$

where D is a domain with a boundary B. It can be recast into the theory of self-adjoint operator $\mathscr{L}$ simply as

$$\langle \mathscr{L} u | v \rangle = \langle u | \mathscr{L} v \rangle \tag{2}$$

where u and v are two elasticity systems and

$$\mathscr{L} = \begin{bmatrix} D_{ij} & 0 \\ 0 & -B_{ij} \end{bmatrix} \tag{3}$$

If the material is elastic and isotropic, the operator D_{ij} can be expressed explicitly as

$$D_{ij} = (\lambda + G)\partial_i \partial_j + G\delta_{ij}\partial_k \partial_k \tag{4}$$

while B_{ij} is the traction operator defined by

$$B_{ij} = \lambda n_i \partial_j + G(n_j \partial_i + \delta_{ij} n_k \partial_k) \tag{5}$$

where λ and G are Lame's constants; n_i is the direction cosines of unit outward normal to the boundary; δ_{ij} denotes Kronecker delta symbol; and ∂_i is the partial differential operator. Note that the equations of equilibrium for the two states, u_j and u_j^*, are

$$D_{ij}u_j(x) + b_i(x) = 0, \quad x \in D \tag{6}$$

$$D_{ij}u_j^*(x) + b_i^*(x) = 0, \quad x \in D \tag{7}$$

with the Cauchy formula

$$B_{ij}u_j(x) = t_i(x), \quad x \text{ on } B \tag{8}$$

$$B_{ij}u_j^*(x) = t_i^*(x), \quad x \text{ on } B \tag{9}$$

By choosing

$$u_j^* = v_j \tag{10}$$

we can obtain

$$\int_D (v_i D_{ij} u_j - u_i D_{ij} v_j)\, \mathrm{d}V = \int_B (v_i B_{ij} u_j - u_i B_{ij} v_j)\, \mathrm{d}B \tag{11}$$

Now we choose specifically,

$$u_i^*(x) = v_i(x) = U_{ij}(x,s)e_j^*(s) \tag{12}$$

$$t_i^*(x) = B_{ik}v_k(x) = B_{ik}(x)U_{kj}(x,s)e_j^*(s) = T_{ij}(x,s)e_j^*(s) \tag{13}$$

$$b_i^*(x) = -D_{ik}(x)v_k(x) = -D_{ik}(x)U_{kj}(x,s)e_j^*(s) = \delta_{ij}(x,s)e_j^*(s) \tag{14}$$

where $U_{ij}(x,s)$ and $T_{ij}(x,s)$ are the Kelvin free-space Green's functions (or fundamental solutions) of the ith direction responses for displacement and traction at the point x, respectively, due to a concentrated load in the jth direction at the point s; and $e_j^*(s)$ is an arbitrary unit-concentrated load at the point s. Then we have Somigliana's identity [13, 14]:

$$\int_B [U_{ij}(x,s)t_i(x) - T_{ij}(x,s)u_i(x)]\, \mathrm{d}B(x) = \begin{cases} u_j(s), & s \in D \\ 0, & s \notin D \end{cases} \tag{15}$$

By changing x and s, Equation (15) is changed to

$$\int_B [U_{ki}(s,x)t_k(s) - T_{ki}(s,x)u_k(s)]\, \mathrm{d}B(s) = \begin{cases} u_i(x), & x \in D \\ 0, & x \notin D \end{cases} \tag{16}$$

In deriving Equation (15), we have omitted the unit vector e_j^* from both sides of the equation because of its arbitrariness. In order to have an additional and independent equation for the problem with a degenerate boundary or degenerate scale, we apply the traction operator B_{pi} to Equation (16) and define

$$B_{pi}(x)\{U_{ki}(s,x)\} = L_{kp}(s,x) \tag{17}$$

$$B_{pi}(x)\{T_{ki}(s,x)\} = M_{kp}(s,x) \tag{18}$$

It then follows that

$$\int_B [L_{kp}(s,x)t_k(s) - M_{kp}(s,x)u_k(s)]\,\mathrm{d}B(s) = \begin{cases} t_p(x), & x \in D \\ 0, & x \notin D \end{cases} \tag{19}$$

Equations (16) and (19) are termed the dual boundary integral equations [13] for the point x in the domain. It is noted that this definition is quite different from that defined by Buecker [15]. A detailed discussion for the dual boundary integral equations can be found in the review article of Chen and Hong [14].

3. DERIVATIONS OF DUAL BOUNDARY INTEGRAL EQUATIONS FOR THE BOUNDARY POINTS

Equations (16) and (19) are derived for a point in the interior domain. By moving the point to the boundary, we are immediately confronted with the problem of singularities and improper integrals. Equation (16) reduces to

$$\int_B U_{ki}(s,x)t_k(s)\,\mathrm{d}B(s) + \beta_{ij}u_j(x) - CPV\int_B T_{ki}(s,x)u_k(s)\,\mathrm{d}B(s) = \delta_{ij}u_j(x) \tag{20}$$

where β_{ij} depends on the non-smooth boundary and CPV denotes the Cauchy principal value. Similarly, Equation (16) becomes

$$\int_B U_{ki}(s,x)t_k(s)\,\mathrm{d}B(s) + (-\delta_{ij} + \beta_{ij})u_j(x) - CPV\int_B T_{ki}(s,x)u_k(s)\,\mathrm{d}B(s) = 0 \tag{21}$$

where β_{ij} reduces to $1/2\delta_{ij}$ when x is on the smooth boundary [16, 17]. Equation (20) reduces to

$$\frac{1}{2}u_i(x) = \int_B U_{ki}(s,x)t_k(s)\,\mathrm{d}B(s) - CPV\int_B T_{ki}(s,x)u_k(s)\,\mathrm{d}B(s), \quad x \text{ on } B \tag{22}$$

Now applying the traction operator to Equation (22), and noting that

$$\begin{aligned} B_{pi}(x)\left\{\int_B U_{ki}(s,x)t_k(s)\,\mathrm{d}B(s)\right\} &= B_{pi}(x)\int_{B-B_\varepsilon} U_{ki}(s,x)t_k(s)\,\mathrm{d}B(s) \\ &= CPV\int_B L_{kp}(s,x)t_k(s)\,\mathrm{d}B(s) \end{aligned} \tag{23}$$

where the first equality results from the integral over the small detour around $x \in B_\varepsilon$, and B_ε denotes a small spherical or circular detour of vanishing radius ε with centre at x. The second equality stems from the boundary terms due to the traction operator using Leibnitz' rule cancelling themselves out, and defining that

$$B_{pi}(x)\left\{CPV\int_B T_{ki}(s,x)u_k(s)\,\mathrm{d}B(s)\right\}\equiv HPV\int_B M_{kp}(s,x)u_k(s)\,\mathrm{d}B(s) \tag{24}$$

we have

$$\frac{1}{2}t_p(x)=CPV\int_B L_{kp}(s,x)t_k(s)\,\mathrm{d}B(s)-HPV\int_B M_{kp}(s,x)u_k(s)\,\mathrm{d}B(s),\quad x\text{ on }B \tag{25}$$

where HPV denotes the Hadamard principal value [18]. Equations (22) and (25) are termed the dual boundary integral equations for a boundary point.

4. DUAL BOUNDARY ELEMENT FORMULATION FOR ELASTICITY

By discretizing the boundary B into constant elements in Equations (22) and (25), we have

$$\frac{1}{2}u_i(x)=\sum_{l=1}^{N}\int_{B_l}U_{ki}(s,x)\,\mathrm{d}B(s)t_k(s_l)-\sum_{l=1}^{N}CPV\int_{B_l}T_{ki}(s,x)\,\mathrm{d}B(s)u_k(s_l) \tag{26}$$

$$\frac{1}{2}t_i(x)=\sum_{l=1}^{N}CPV\int_{B_l}L_{ki}(s,x)\,\mathrm{d}B(s)t_k(s_l)-\sum_{l=1}^{N}HPV\int_{B_l}M_{ki}(s,x)\,\mathrm{d}B(s)u_k(s_l) \tag{27}$$

where N is the number of boundary elements and B_l is the lth boundary element. For a two-dimensional problem, Equations (26) and (27) can be written in matrix forms as shown below:

$$[C]_{2N\times 2N}\{u\}_{2N\times 1}=[U]_{2N\times 2N}\{t\}_{2N\times 1}-[T]_{2N\times 2N}\{u\}_{2N\times 1} \tag{28}$$

$$[C]_{2N\times 2N}\{t\}_{2N\times 1}=[L]_{2N\times 2N}\{t\}_{2N\times 1}-[M]_{2N\times 2N}\{u\}_{2N\times 1} \tag{29}$$

where $\{u\}$ and $\{t\}$ are the column vectors of boundary displacement and traction, and $[C]$ is a matrix of free terms. Equations (28) and (29) reduce to

$$[\bar{T}]\{u\}=[U]\{t\} \tag{30}$$

$$[M]\{u\}=[\bar{L}]\{t\} \tag{31}$$

where $[\bar{T}]$ and $[\bar{L}]$ differ from $[T]$ and $[L]$ by a matrix of free terms. It is now well known that Equation (30) is not sufficient to provide enough constraint equations for crack problems; thus, Equation (31) is needed. Combining Equations (30) with (31), the double unknowns on the degenerate boundary could be determined easily. For displacement specified problems with certain geometry, the $[U]$ matrix may be singular due to the degenerate scale. In this paper, we will examine the role of Equation (31) in dealing with the degenerate scale problems.

5. MATHEMATICAL ANALYSIS OF THE DEGENERATE SCALE PROBLEMS FOR AN ELASTICITY PROBLEM USING POTENTIAL THEORY

We propose a theoretical approach to understand the mechanism of degenerate scale. If we distribute the single-layer potential $\Phi_k(s)$ the boundary B, we have

$$u_i(x) = \int_B U_{ik}(s,x)\Phi_k(s)\,\mathrm{d}B(s), \quad x \in D \tag{32}$$

where the single-layer density $\Phi_k(s)$ is expressed by

$$\Phi_k(s) = \sum_{j=0}^{N} c_{jk}\Phi_k^{(j)}(s), \quad k = 1 \text{ or } 2, \; s \in B \tag{33}$$

Similarly, we have

$$u_i(x) = \int_B T_{ik}(s,x)\Psi_k(s)\,\mathrm{d}B(s), \quad x \in D \tag{34}$$

where the double-layer potential $\Psi_k(s)$ is represented by

$$\Psi_k(s) = \sum_{j=0}^{N} d_{jk}\Psi_k^{(j)}(s), \quad k = 1 \text{ or } 2, \; s \text{ on } B \tag{35}$$

The primary field potential across the boundary is continuous in Equation (32) and discontinuous in Equation (34). The secondary field across the boundary is discontinuous in Equation (32) and pseudo-continuous in Equation (34). If either one component of the displacement field resulted from the $U(T)$ kernel is equal to zero everywhere in the interior domain due to a special distribution $\Phi_k(s)$ ($\Psi_k(s)$), the geometry scale is degenerate. Therefore, the strength of this singularity distribution, c_{jk} (d_{jk}), cannot be determined. Then we can find the degenerate scale boundary for the problem with corresponding boundary distribution $\Phi_k(s)$ ($\Psi_k(s)$).

Now, we need to find a boundary distribution Φ_k with a non-zero norm and have the potential distribution

$$u_i(x) = \text{constant} = \int_B U_{ik}(x,s)\Phi_k(s)\,\mathrm{d}B(s), \quad x \in B \tag{36}$$

and

$$t_i(x) = 0 = \int_B L_{ik}(x,s)\Phi_k(s)\,\mathrm{d}B(s), \quad x \in B \tag{37}$$

Based on the maximum principle for elasticity, the maximum value must occur on the boundary. If the size of the special geometry scale causes the constant in Equation (36) to be zero, the special geometry dimension is a degenerate scale.

6. MATHEMATICAL ANALYSIS OF THE DEGENERATE SCALE PROBLEMS FOR AN ELLIPTIC-DOMAIN PROBLEM IN ELASTICITY USING STRESS FUNCTION

For a problem of plane elasticity, the governing equation of Equation (6) is the Navier's equation

$$(\lambda + G)\nabla(\nabla \cdot \mathbf{u}(x)) + G\nabla^2\mathbf{u}(x) = 0, \quad x \in D \tag{38}$$

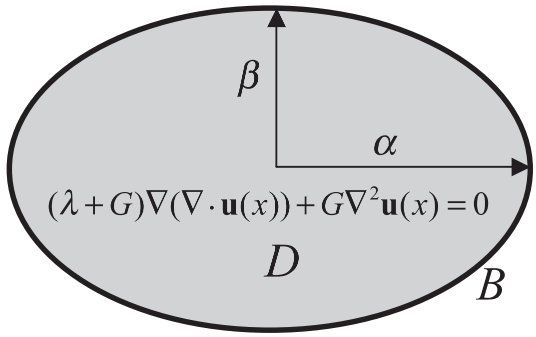

Figure 1. Plane elasticity problem with an elliptic region.

where $\mathbf{u}(x)$ is the displacement vector of (u_1, u_2). By introducing the Airy stress function [19], ϕ, satisfying the biharmonic equation

$$\nabla^2(\nabla^2\phi)=0 \tag{39}$$

the general solution of Equation (39) can be represented by

$$\phi=\mathrm{Re}[\bar{z}\psi(z)+\chi(z)] \tag{40}$$

where the bar, '¯', is the complex conjugate, Re denotes the real part, $\psi(z)$ and $\chi(z)$ can be any analytic functions of the complex variable z, $z=x+y\mathrm{i}$. When $\phi(z)$ and $\chi(z)$ are both known, we can obtain the displacement in plane stress from

$$2G(u_1+\mathrm{i}u_2)=\frac{3-v}{1+v}\,\psi(z)-z\bar{\psi}'(\bar{z})-\bar{\chi}'(\bar{z}) \tag{41}$$

where u_1 and u_2 are the displacements in the x and y directions and v is the Poisson's ratio.

We consider an infinite plate as shown in Figure 1 and use the elliptic co-ordinates ξ and η defined by

$$z=k\cosh\zeta,\quad \zeta=\xi+\mathrm{i}\eta \tag{42}$$

which gives

$$x=k\cosh\xi\cos\eta,\quad y=k\sinh\xi\sin\eta \tag{43}$$

The co-ordinate ξ is a constant and is equal to ξ_0 on an ellipse of semi-axes $k\cosh\xi_0$ and $k\sinh\xi_0$. If the two semi-axes are given as α and β, the values k and ξ_0 can be determined by

$$k\cosh\xi_0=\alpha,\quad k\sinh\xi_0=\beta \tag{44}$$

We can choose $\chi(z)$ and $\psi(z)$ in the interior and exterior domains as

$$\psi^{(\mathrm{i})}(z)=0 \tag{45}$$

$$\chi^{(\mathrm{i})}(z)=d_1k\cosh\zeta \tag{46}$$

and

$$\psi^{(e)}(z) = a_1\zeta \tag{47}$$

$$\chi^{(e)}(z) = b_1 k(\zeta\cosh\zeta - \sinh\zeta) + b_2 k\sinh(2\xi_0 - \zeta) - b_3 k\cosh\zeta \tag{48}$$

Substituting Equations (45)–(48) into Equation (41), we have

$$2G(u_1^{(i)} + \mathrm{i}u_2^{(i)}) = -\bar{d}_1 \tag{49}$$

$$2G(u_1^{(e)} + \mathrm{i}u_2^{(e)}) = a_1\frac{3-v}{1+v}\zeta - \bar{a}_1\frac{\cosh\zeta}{\sinh\bar{\zeta}} - \bar{b}_1\bar{\zeta} + \bar{b}_2\frac{\cosh(2\xi_0 - \bar{\zeta})}{\sinh\bar{\zeta}} + \bar{b}_3 \tag{50}$$

where the subscripts '(i)' and '(e)' denote the interior point and the exterior point, respectively, and a_1, b_1, b_2, b_3 and c_1 are the complex constants. For the continuity of displacement across the boundary, the displacement by approaching from the exterior domain must be equal to that by approaching from the interior domain to the boundary ($\zeta = \xi_0 + \mathrm{i}\eta$). The coefficients can be chosen as $\bar{b}_1 = -(3-v)/(1+v)a_1$ and $b_2 = a_1$. The displacement in the exterior domain becomes

$$2G(u_1^{(e)} + \mathrm{i}u_2^{(e)}) = \frac{3-v}{1+v}(\zeta + \bar{\zeta})a_1 + \frac{\cosh(2\xi_0 - \bar{\zeta}) - \cosh\zeta}{\sinh\bar{\zeta}}\bar{a}_1 + \bar{b}_3 \tag{51}$$

When ξ approaches to infinity, the displacement approaches to $\ln r$ and we have the asymptotic form

$$2G(u_1^{(e)} + \mathrm{i}u_2^{(e)}) \simeq \frac{2(3-v)}{1+v}\left(\ln r - \ln\frac{k}{2}\right)a_1 - \mathrm{e}^{2\mathrm{i}\theta}\bar{a}_1 + \mathrm{e}^{-2\xi_0} + \bar{b}_3 \tag{52}$$

where $x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}$, and

$$\begin{aligned}\bar{b}_3 &= \frac{2(3-v)}{1+v}\ln\frac{k}{2}a_1 - \mathrm{e}^{-2\xi_0}\bar{a}_1\\ &= \frac{3-v}{1+v}[\ln(\alpha^2+\beta^2) - \ln 4]a_1 - \frac{\alpha-\beta}{\alpha+\beta}\bar{a}_1\end{aligned} \tag{53}$$

When ξ approaches ξ_0 on the elliptic boundary, we have

$$\begin{aligned}2G(u_1^{(e)} + \mathrm{i}u_2^{(e)}) &= \frac{3-v}{1+v}[2\xi_0 + \ln(\alpha^2+\beta^2) - \ln 4]a_1 - \frac{\alpha-\beta}{\alpha+\beta}\bar{a}_1\\ &= \frac{2(3-v)}{1+v}\ln\frac{\alpha+\beta}{2}a_1 - \frac{\alpha-\beta}{\alpha+\beta}\bar{a}_1\end{aligned} \tag{54}$$

On the boundary, the displacement near the interior domain is equal to the potential near the exterior domain for continuity requirement. From Equations (49) and (54), we have

$$-\bar{d}_1 = \frac{2(3-v)}{1+v}\ln\frac{\alpha+\beta}{2}a_1 - \frac{\alpha-\beta}{\alpha+\beta}\bar{a}_1 \tag{55}$$

Alternatively, Equation (49) can be written as

$$2Gu_1^{(\mathrm{i})} = \left\{ \frac{2(3-v)}{1+v} \ln \frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{\alpha+\beta} \right\} \mathrm{Re}[a_1] \tag{56}$$

$$2Gu_2^{(\mathrm{i})} = \left\{ \frac{2(3-v)}{1+v} \ln \frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{\alpha+\beta} \right\} \mathrm{Im}[a_1] \tag{57}$$

where Im denote the imaginary part. We find that either one component of the displacement field in the interior domain is null when either $(2(3-v))/(1+v)\ln(\alpha+\beta)/2-(\alpha-\beta)/(\alpha+\beta)=0$ or $(2(3-v))/(1+v)\ \ln(\alpha+\beta)/2+(\alpha-\beta)/(\alpha+\beta)=0$. This indicates that degenerate scale occurs when

$$2\kappa \ln \frac{\alpha+\beta}{2} = \frac{\alpha-\beta}{\alpha+\beta} \tag{58}$$

or

$$2\kappa\ \ln \frac{\alpha+\beta}{2} = -\frac{\alpha-\beta}{\alpha+\beta} \tag{59}$$

where $\kappa=\frac{3-v}{1+v}$.

If β is specified to be $\chi\alpha$, the degenerate scale in Equations (58) and (59) can be rewritten in the following forms:

$$\alpha = \frac{2}{1+\chi}\, \mathrm{e}^{(1/2\kappa)(1-\chi)/(1+\chi)} \tag{60}$$

or

$$\alpha = \frac{2}{1+\chi}\, \mathrm{e}^{(1/2\kappa)(\chi-1)/(\chi+1)} \tag{61}$$

For the plane strain case, v should be replaced by $v/(1-v)$. For the circular domain, $\chi=1$ and $\alpha=\beta$, the degenerate scale reduces to radius of one. Both components of the displacement vanish for the boundary distribution when the degenerate scale occurs.

7. NUMERICAL EXAMPLES FOR PROBLEMS WITH CIRCULAR AND ELLIPTIC DOMAINS

In Figure 2, we consider the interior problem of plane elasticity (plane strain) with a circular domain where the displacement is specified on the boundary. The shear modulus is $G=1.0\,\mathrm{N/m^2}$ and Poisson's ratio is $v=0.25$. In the BEM mesh, 10 elements are distributed uniformly on the boundary. We decompose the $[U]$ and $[L]$ matrices by using the singular value decomposition (SVD) technique and plot the minimum singular values versus the radius a as shown in Figures 3 and 4. We find that the $[U]$ matrix is singular when the radius is equal to one as shown in Figure 3 since the degenerate scale was proved theoretically at $a=1.0$. Nevertheless, the $[L]$ matrix is never singular whatever the radius a is. Figure 3 indicates that hypersingular formulation (LM equation) can shift the zero singular value in the singular formulation (UT equation).

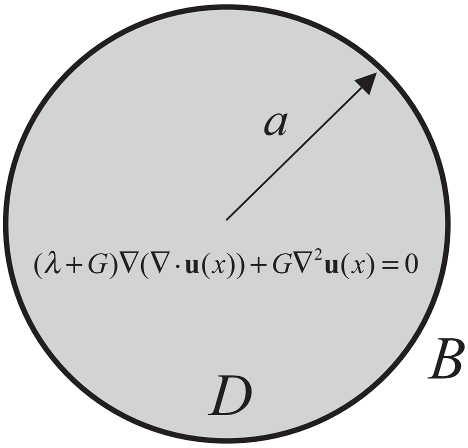

Figure 2. Plane elasticity problem with a circular domain.

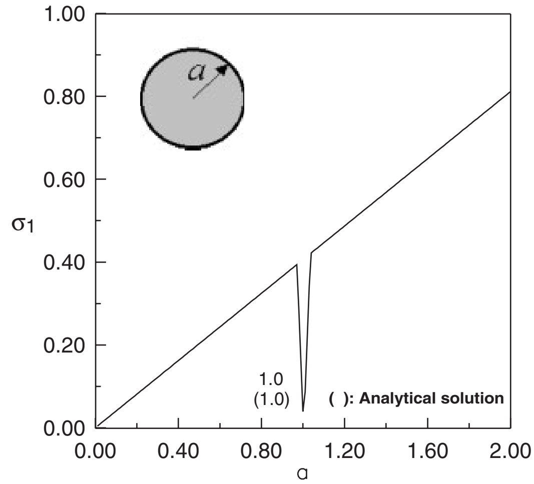

Figure 3. The minimum singular value σ_1 of $[U]$ matrix versus radius a for a plane elasticity problem with a circular domain.

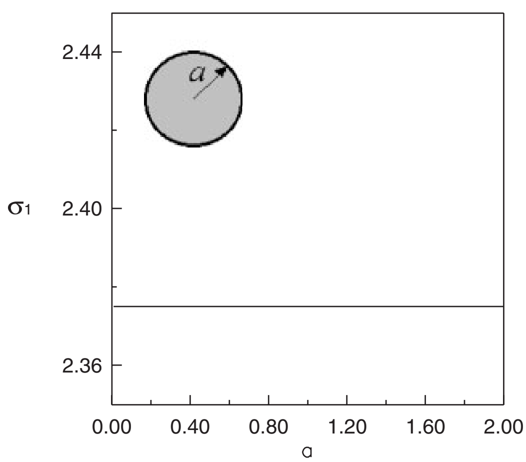

Figure 4. The minimum singular value σ_1 of $[L]$ matrix versus radius a for a plane elasticity problem with a circular domain.

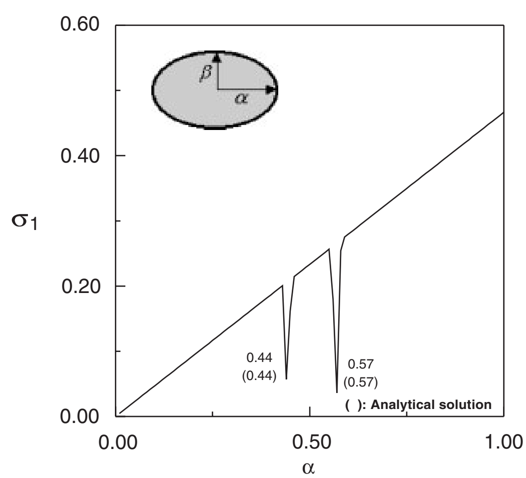

Figure 5. The minimum singular value σ_1 of $[U]$ matrix versus semi-axes α for a plane elasticity problem with an elliptic domain ($v = 0.25$).

As shown in Figure 1, we consider an elliptic problem (plane strain) with the semi-axes, α and β, where $\beta = 3\alpha$. The displacement is specified on the boundary. The shear modulus is $G = 1.0\,\mathrm{N/m^2}$ and Poisson's ratio is $v = 0.25$. In the BEM mesh, ten elements are distributed uniformly on the boundary. We determined the minimum singular value of the matrices $[U]$ and $[L]$ by using the SVD technique. The minimum singular values of $[U]$ and $[L]$ matrices versus the geometry scale are plotted in Figures 5 and 6, respectively. The degenerate scale

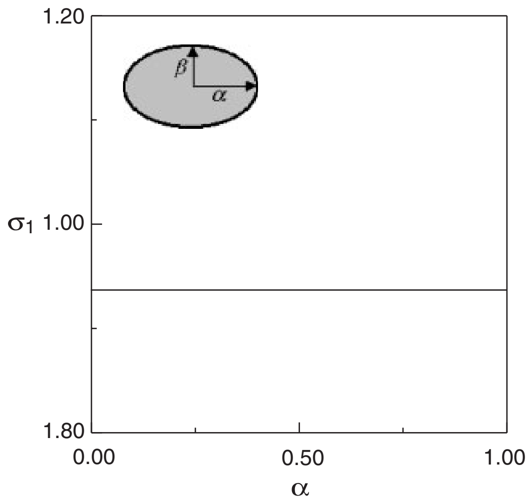

Figure 6. The minimum singular value σ_1 of $[L]$ versus semi-axes α for a plane elasticity problem with an elliptic domain ($v = 0.25$).

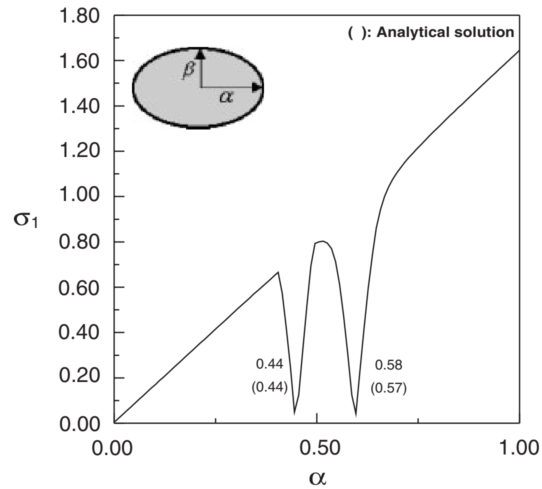

Figure 7. The minimum singular value σ_1 of $[U]$ matrix versus semi-axes α for a plane elasticity problem with an elliptic domain ($v = 0.30$).

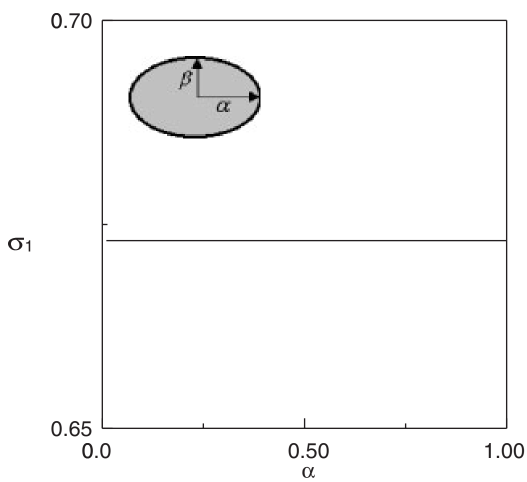

Figure 8. The minimum singular value σ_1 of $[L]$ matrix versus semi-axes α for a plane elasticity problem with an elliptic domain ($v = 0.30$).

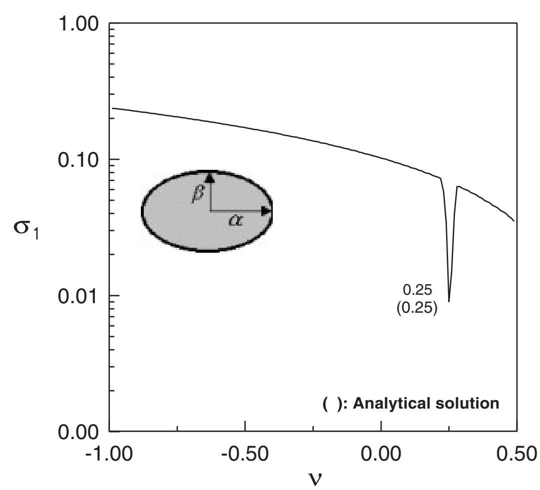

Figure 9. The minimum singular value σ_1 of $[U]$ matrix versus Poisson's v for the elliptic domain problem with semi-axes $\alpha = 1.323$ and $\beta = 0.441$.

occurs at $\alpha = 0.441$ or 0.567. It is found that the conventional BEM ($[U]$ matrix) has the degenerate scale at $\alpha = 0.44$ and 0.57 as predicted theoretically in Equations (60) and (61). Nevertheless, the $[L]$ matrix is never singular for any geometry scale.

In order to know how the Poisson's ratio influences the degenerate scale, we consider the above problem with a different Poisson's ratio of 0.3. The first singular values of $[U]$ and $[L]$

matrices versus the geometry scale are plotted in Figures 7 and 8. The degenerate scale occurs at $\alpha = 0.435$ or 0.575. It is found that the conventional BEM results in the degenerate scale at $\alpha = 0.44$ and 0.58 as theoretically predicted in Equations (60) and (61). The $[L]$ formulation can remove the zero singular value in the case of degenerate scale of $[U]$ matrix.

Finally, we consider the elliptic plane elasticity problem (plane strain). The shear modulus is $G = 1.0\,\mathrm{N/m^2}$ and semi-axes are $\alpha = 1.323$ and $\beta = 0.414$. In the BEM mesh, 36 elements are distributed uniformly on the boundary. Figure 9 shows the minimum singular value of matrix $[U]$ versus Poisson's ratio v. For this problem, the matrix is singular at $v = 0.25$ analytically. The results of the dual BEM and mathematical analysis are in good agreement.

8. CONCLUSIONS

In this paper, we have proven how the degenerate scale occurs in the BEM for two-dimensional elasticity problems in elliptical domains. Based on the stress function formulation, a singularity distribution along a boundary of degenerate scale is found to have a null field in the interior domain. For the circular domain problem, the radius of one is a degenerate scale if the singular integral equation is used. To overcome the problem, a hypersingular equation can be adopted. In case of the elliptic domain problem, Equations (60) and (61) are the criteria where degenerate scale appears if the singular integral equation is used. The numerical examples have shown that the degenerate scale depends on the geometry scale and the Poisson's ratio in elasticity problems. Numerical results using the dual BEM agree well with mathematical prediction using stress function.

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