In the course, we have derived the constraint among $u(0), u(1), t(0)$ and $t(1)$ by using the integral equation for the domain point approaching boundary $(s=0^+$ and $s=1^-)$ for UT and LM formulations.

Please derive the constraint by the null-field integral equations approaching boundary $(s=0^-$ and $s=1^+)$ for UT and LM formulations.

ANS 欲解系統: $\dfrac{d^2u(x)}{d^2x}=0,\quad 0<x<1 \quad u(0)=a, u(1)=b$

輔助系統: $\dfrac{d^2U(x,s)}{dx^2}=\delta|x-s| \quad \rightarrow U(x,s)=\dfrac{1}{2}|x-s|$

$$\rightarrow U(s,x)=\begin{cases}\frac{1}{2}(x-s) & x>s \\ \frac{1}{2}(s-x) & x<s\end{cases} \qquad T(s,x)=\begin{cases}-\frac{1}{2} & x>s \\ \frac{1}{2} & x<s\end{cases}$$

$$\rightarrow L(s,x)=\begin{cases}\frac{1}{2} & x>s \\ -\frac{1}{2} & x<s\end{cases} \qquad M(s,x)=\begin{cases}0 & x>s \\ 0 & x<s\end{cases}$$

積分方程表示式: $u(x)=[\dfrac{dU(s,x)}{ds}u(s)-\dfrac{du(s)}{ds}U(s,x)]\Big|_0^1$

$$\rightarrow u(x)=[T(s,x)u(s)-\frac{du(s)}{ds}U(s,x)]\Big|_0^1$$

(1) As $x=0^-$

$$0=[T(s,0^-)u(s)-\frac{du(s)}{ds}U(s,0^-)]\Big|_0^1$$

$$\rightarrow 0=[T(1,0^-)u(1)-\frac{du(1)}{ds}U(1,0^-)]-[T(0,0^-)u(0)-\frac{du(0)}{ds}U(0,0^-)]$$

$$\rightarrow 0=\frac{1}{2}u(1)-\frac{1}{2}\frac{du(1)}{ds}-\frac{1}{2}u(0)$$

$$\rightarrow \frac{1}{2}b-\frac{1}{2}a=\frac{1}{2}\frac{du(1)}{ds} \qquad \rightarrow \frac{du(1)}{ds}=t(1)=b-a$$

(2) As $x=1^+$

$$0=[T(s,1^+)u(s)-\frac{du(s)}{ds}U(s,1^+)]\Big|_0^1$$

$$\rightarrow 0=[T(1,1^+)u(1)-\frac{du(1)}{ds}U(1,1^+)]-[T(0,1^+)u(0)-\frac{du(0)}{ds}U(0,1^+)]$$

$$\rightarrow 0=-\frac{1}{2}u(1)+\frac{1}{2}u(0)+\frac{1}{2}\frac{du(0)}{ds}$$

$$\rightarrow \frac{1}{2}b-\frac{1}{2}a=\frac{1}{2}\frac{du(0)}{ds} \quad \rightarrow \frac{du(0)}{ds}=t(0)=b-a$$

將 $t(0)=b-a$， $t(1)=b-a$ 代入原式 $u(x)=[T(s,x)u(s)-\frac{du(s)}{ds}U(s,x)]\Big|_0^1$

$$\rightarrow u(x)=T(1,x)u(1)-\frac{du(1)}{ds}U(1,x)-T(0,x)u(0)-\frac{du(0)}{ds}U(0,x)$$

$$=\frac{1}{2}b-\frac{1}{2}(1-x)(b-a)+\frac{1}{2}a+\frac{1}{2}x(b-a)=x(b-a)+a$$

積分方程表示式: $u^{'}(x)=[\frac{\partial T(s,x)}{\partial x}u(s)-\frac{\partial U(s,x)}{\partial x}t(s)]\Big|_0^1$

$$u^{'}(x)=[M(s,x)u(s)-L(s,x)t(s)]\Big|_0^1$$

(1) As $x=0^-$

$$0=-L(s,0^-)t(s)\Big|_0^1$$

$$\rightarrow 0=-L(1,0^-)t(1)+L(0,0^-)t(0)$$

$$=\frac{1}{2}u^{'}(1)-\frac{1}{2}u^{'}(0)$$

$$\rightarrow u^{'}(1)=u^{'}(0) \text{———————————————— } 1$$

(2) As $x=1^+$

$$0=-L(s,1^+)t(s)\Big|_0^1$$

$$\rightarrow 0=-L(1,1^+)t(1)+L(0,1^+)t(0)$$

$$=-\frac{1}{2}u^{'}(1)+\frac{1}{2}u^{'}(0)$$

$$\rightarrow u^{'}(1)=u^{'}(0) \text{———————————————— } 2$$

1,2 式相依，故無法求解 $u^{'}(1), u^{'}(0)$