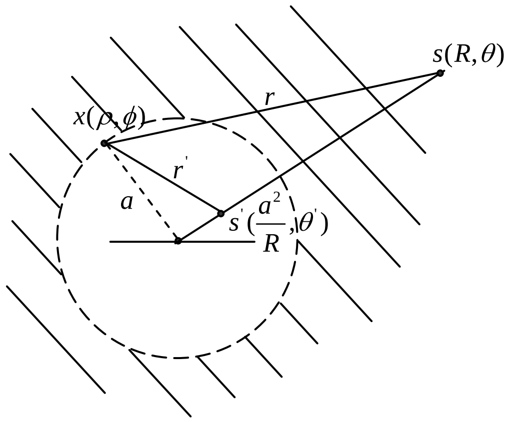
1. $\nabla^2 G(x,s) = 2\pi\delta(x-s) \quad G(x,s)=0, \quad x \in B$



$$\ln r = \ln|x-s| = \ln R - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho}{R}\right)^m \cos[m(\theta-\phi)], \quad \rho < R$$

$$\ln r' = \ln|x-s'| = \ln \rho - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R'}{\rho}\right)^m \cos[m(\theta'-\phi)], \quad \rho > R$$

$$\because G(x,s)=0,\ x\in B \quad \rho = a \qquad \therefore \theta=\theta' \quad \frac{\rho}{R}=\frac{R'}{\rho} \quad \rightarrow RR' = a^2$$

$$G(x,s) = \ln|x-s| - \ln|x-s'| + \ln a - \ln R$$

$$2\pi u(s) = -\int_0^{2\pi}\frac{\partial G(x,s)}{\partial n_x}u(\phi)a\,d\phi = -\int_0^{2\pi}\frac{\partial G(x,s)}{\partial \rho}u(\phi)a\,d\phi$$

$$\rightarrow -\int_0^{2\pi}\frac{\partial}{\partial\rho}[\ln|x-s| - \ln|x-s'|]u(\phi)a\,d\phi$$

$$2R\rho\cos(\phi-\theta) = \rho^2 + R^2 - r^2 \quad \rightarrow r = \sqrt{\rho^2 + R^2 - 2R\rho\cos(\phi-\theta)}$$

$$2\left(\frac{a^2}{R}\right)\rho\cos(\phi-\theta) = \rho^2 + \left(\frac{a^2}{R}\right)^2 - r'^2 \quad \rightarrow r' = \sqrt{\rho^2 + \frac{a^4}{R^2} - 2\frac{\rho a^2}{R}\cos(\phi-\theta)}$$

$$\frac{\partial r}{\partial\rho}\Big|_{\rho=a} = \frac{1}{2}\frac{2a - 2R\cos(\phi-\theta)}{r}, \quad \frac{\partial r'}{\partial\rho}\Big|_{\rho=a} = \frac{1}{2}\frac{2a - 2\frac{a^2}{R}\cos(\phi-\theta)}{r'}$$

$$\frac{\partial}{\partial\rho}\left(\ln\frac{r}{r'}\right) = \frac{1}{r}\frac{\partial r}{\partial\rho} - \frac{1}{r'}\frac{\partial r'}{\partial\rho}$$

$$\rightarrow 2\pi u(s) = -\int_0^{2\pi}\frac{\partial}{\partial\rho}[\ln r - \ln r']u(\phi)a\,d\phi = -\int_0^{2\pi}\frac{a^2-R^2}{R^2+a^2-2\rho R\cos(\phi-\theta)}u(\phi)\,d\phi$$

$x \rightarrow s; \quad s \rightarrow x$

$$\rightarrow u(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\frac{a^2-R^2}{a^2+R^2-2R\rho\cos(\theta-\phi)}u(\theta)\,d\theta$$

$$u(p,\phi)=\frac{1}{2\pi}\int_0^{2\pi}\frac{p^2-1}{p^2+1-2p\cos\theta}d\theta$$

2. I. $\oint\frac{1}{(1-2p\cos\theta+p^2)}d\theta;|p|>1$

$$\nabla^2u(p,\phi)=0,u(1,\phi)=1\quad\to u(p,\phi)=1$$

$$1=\frac{1}{2\pi}\int_0^{2\pi}\frac{p^2-1}{p^2+1-2p\cos\theta}d\theta\quad\to\int_0^{2\pi}\frac{1}{p^2+1-2p\cos\theta}d\theta=\frac{2\pi}{p^2-1}$$

II. $\oint\frac{\cos(2\theta)}{(1-2p\cos\theta+p^2)}d\theta;|p|>1$

$$\nabla^2u(p,\phi)=0,u(1,\phi)=\cos(2\phi)\quad\to u(p,\phi)=p^{-2}\cos(2\phi)$$

$$p^{-2}\cos(2\phi)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(p^2-1)\cos(2\theta)}{p^2+1-2p\cos\theta}d\theta\quad\to\int_0^{2\pi}\frac{\cos(2\theta)}{p^2+1-2p\cos\theta}d\theta=\frac{2\pi}{p^4-p^2}$$

III. $\oint\frac{1-p\cos\theta}{(1-2p\cos\theta+p^2)}d\theta;|p|>1$

$$\to\oint\frac{1}{(1-2p\cos\theta+p^2)}-\frac{p\cos\theta}{(1-2p\cos\theta+p^2)}d\theta;|p|>1$$

$$\nabla^2u(p,\phi)=0,u(1,\phi)=\cos(\phi)\quad\to u(p,\phi)=p\cos(\phi)$$

$$\frac{\cos(\phi)}{p}=\frac{1}{2\pi}\int_0^{2\pi}\frac{(p^2-1)\cos(\theta)}{p^2+1-2p\cos\theta}d\theta\quad\to\int_0^{2\pi}\frac{\cos(\theta)}{p^2+1-2p\cos\theta}d\theta=\frac{2\pi}{p^3-p}$$

$$\to\oint\frac{1}{(1-2p\cos\theta+p^2)}-\frac{p\cos\theta}{(1-2p\cos\theta+p^2)}d\theta;|p|>1$$
$$=\frac{2\pi}{p^2-1}-\frac{2\pi}{p^2-1}=0$$

IV. $\oint\frac{\sin\theta}{(1-2p\cos\theta+p^2)}d\theta;|p|>1$

$$\nabla^2u(p,\phi)=0,u(1,\phi)=\sin(\phi)\quad\to u(p,\phi)=p\sin(\phi)$$

$$\frac{\sin(\phi)}{p}=\frac{1}{2\pi}\int_0^{2\pi}\frac{(p^2-1)\sin(\theta)}{p^2+1-2p\cos\theta}d\theta\quad\to\int_0^{2\pi}\frac{\sin(\theta)}{p^2+1-2p\cos\theta}d\theta=0$$

V. $\oint\ln\sqrt{(1-2p\cos(\theta)+p^2)}d\theta,|p|>1$

$$\to\int_0^{2\pi}\int_0^{p}\frac{p-\cos\theta}{1+p^2-2p\cos\theta}dpd\theta=\int_0^{p}\int_0^{2\pi}\frac{p-\cos\theta}{1+p^2-2p\cos\theta}d\theta dp$$

$$f(\theta)=p,u(\rho,\phi)=p$$

$$\int_0^{p}\int_0^{2\pi}\frac{p}{1+p^2-2p\cos\theta}d\theta dp=\int_0^{p}\frac{2\pi p}{p^2-1}dp$$

$$f(\theta)=\cos\theta,u(\rho,\phi)=\frac{1}{p}\cos\phi$$

$$\int_0^p \int_0^{2\pi} \frac{\cos\theta}{1+p^2-2p\cos\theta} d\theta dp = \int_0^p \frac{2\pi}{(1-p^2)p} dp$$

$$\int_0^{2\pi} \ln\sqrt{(1-2p\cos(\theta)+p^2)} d\theta = \int_0^p \frac{2\pi p}{p^2-1} dp - \int_0^p \frac{2\pi}{(1-p^2)p} dp$$

$$= [\pi\ln(p^2-1)]\Big|_0^p - 2\pi[-\ln p + \frac{1}{2}\ln(p^2-1)]\Big|_0^p$$

$$= 2\pi \ln p$$