1.

$$U(s,x)=\begin{cases}\frac{1}{2}(s-x)+c & , \quad s>x\\ \frac{1}{2}(x-s)+c & , \quad x>s\end{cases} \quad , \quad T(s,x)=\begin{cases}-\frac{1}{2} & , \quad s>x\\ \frac{1}{2} & , \quad x>s\end{cases}$$

$$u(x)=\left(u(s)\frac{dU(s,x)}{ds}-U(s,x)\frac{du(s)}{ds}\right)\Bigg|_{s=0}^{s=\ell}=\left[u(s)T(s,x)-U(s,x)\,\acute{u}(s)\right]\Big|_{s=0}^{s=\ell},\quad 0<x<\ell$$

$$x\to 0^- \quad u(0^-)=T(\ell,0^-)u(\ell)-U(\ell,0^-)u'(\ell)-T(0,0^-)u(0)+U(0,0^-)u'(0)$$

$$\to 0=-\frac{q}{2}-\left(\frac{\ell}{2}+c\right)t(\ell)+\frac{p}{2}+ct(0)$$

$$x\to \ell^+ \quad u(\ell^+)=T(\ell,\ell^+)u(\ell)-U(\ell,\ell^+)u'(\ell)-T(0,\ell^+)u(0)+U(0,\ell^+)u'(0)$$

$$\to 0=\frac{q}{2}-ct(\ell)-\frac{p}{2}+\left(\frac{\ell}{2}+c\right)t(0)$$

$$t(\ell)=t(0)=\frac{p-q}{\ell}$$

(a)

$$\begin{pmatrix}U(0,0^-) & -U(\ell,0^-)\\ U(0,\ell^+) & -U(\ell,\ell^+)\end{pmatrix}\begin{Bmatrix}t(0)\\ t(\ell)\end{Bmatrix}=\begin{pmatrix}T(0,0^-) & -T(\ell,0^-)\\ T(0,\ell^+) & -T(\ell,\ell^+)\end{pmatrix}\begin{Bmatrix}u(0)\\ u(\ell)\end{Bmatrix}$$

$$\to\begin{pmatrix}c & -\frac{\ell}{2}-c\\ \frac{\ell}{2}+c & -c\end{pmatrix}\begin{Bmatrix}t(0)\\ t(\ell)\end{Bmatrix}=\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}\begin{Bmatrix}u(0)\\ u(\ell)\end{Bmatrix}$$

(b)

$$\det\begin{pmatrix}0 & -\frac{\ell}{2}\\ \frac{\ell}{2} & 0\end{pmatrix}=\frac{\ell^2}{4}$$

$\frac{\ell^2}{4}$ is not zero, so $\det[U]$ is never zero.

(c)

$$\det\begin{pmatrix}-\frac{1}{4} & -\frac{\ell}{2}+\frac{1}{4}\\ \frac{\ell}{2}-\frac{1}{4} & \frac{1}{4}\end{pmatrix}=0,\quad \left(\frac{\ell}{2}-\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2=0,\quad \ell=0,1$$

2.

(a)

$$p = U(0,-2\ell)P + U(0,-\ell)Q = \ell P + \frac{\ell}{2}Q$$

$$q = U(\ell,-2\ell)P + U(\ell,-\ell)Q = \frac{3\ell}{2}P + \ell Q$$

$$P = \frac{2(2p-q)}{\ell}, \quad Q = \frac{2(-3p+2q)}{\ell}$$

$$u(x) = U(x,-2\ell)P + U(x,-\ell)Q = \frac{(q-p)}{\ell}x + p$$

(b)

$$p = U(0,2\ell)P + U(0,3\ell)Q = \ell P + \frac{3\ell}{2}Q$$

$$q = U(\ell,2\ell)P + U(\ell,3\ell)Q = \frac{\ell}{2}P + \ell Q$$

$$P = \frac{2(2p-3q)}{\ell}, \quad Q = \frac{2(-p+2q)}{\ell}$$

$$u(x) = U(x,2\ell)P + U(x,3\ell)Q = \frac{(q-p)}{\ell}x + p$$

3.

$$u(x,t) = \frac{1}{2p}\int_{-\infty}^{\infty} U(x,w)e^{iwt}dt$$

Assume

$$G(x,s) = \begin{cases} c_1 e^{ikx}, & -\infty < x < s \\ c_2 e^{-ikx}, & s < x < \infty \end{cases}$$

Displacement Continuity at $x = s$

$$c_1 e^{iks} = c_2 e^{-iks}$$

The difference in slope is 1 at $x = s$

$$-ikc_2 e^{-iks} - ikc_1 e^{iks} = 1$$

$$\therefore c_1 = \frac{i}{2k}e^{-iks},\ c_2 = \frac{i}{2k}e^{iks}$$

$$G(x,s) = \begin{cases} \frac{i}{2k}e^{ik(x-s)}, & -\infty < x < s \\ \frac{i}{2k}e^{-ik(x-s)}, & s < x < \infty \end{cases}$$

(a) Wronskian Approach

$$G(x,s) = \begin{cases} c_1 \sin kx, & 0 < x < s \\ c_2 e^{-ikx}, & s < x < \infty \end{cases}$$

Displacement Continuity at $x = s$

$$c_1 \sin ks = c_2 e^{-iks}$$

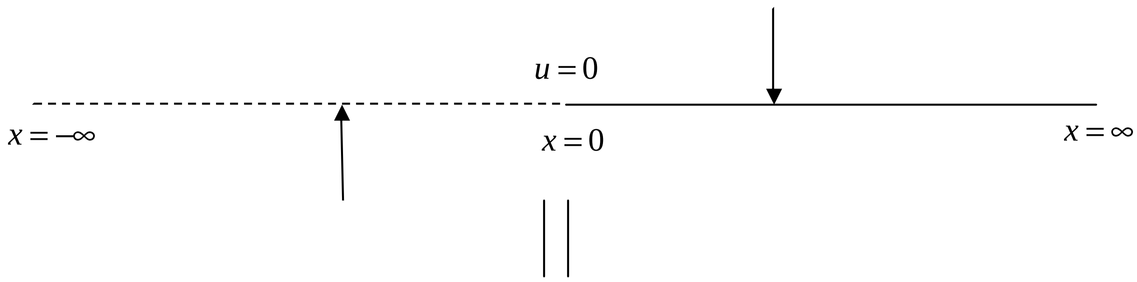
The difference in slope is 1 at $x = s$

$$-ikc_2 e^{-iks} - kc_1 \cos ks = 1$$

$$\therefore c_1 = \frac{-1}{k}e^{-iks},\ \therefore c_2 = \frac{-1}{k}\sin ks$$

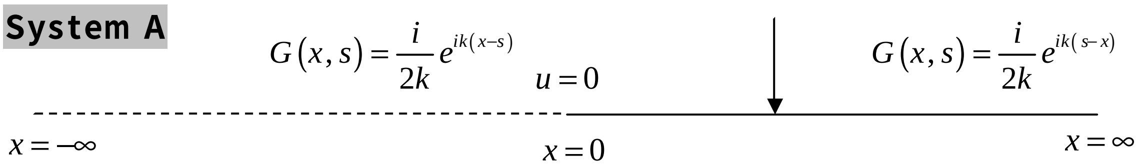
$$G(x,s) = \begin{cases} -\frac{1}{k}e^{-iks}\sin kx, & 0 < x < s \\ -\frac{1}{k}e^{-ikx}\sin ks, & s < x < \infty \end{cases}$$

(a) Image Method



System A

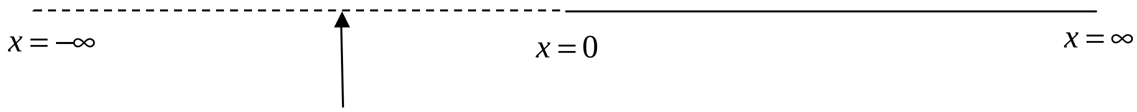
$$G(x,s)=\frac{i}{2k}e^{ik(x-s)} \qquad G(x,s)=\frac{i}{2k}e^{ik(s-x)}$$



+

System B

$$G(x,s)=\frac{-i}{2k}e^{ik(x+s)} \qquad G(x,s)=\frac{-i}{2k}e^{-ik(s+x)}$$



$$G(x,s)=\begin{cases}\dfrac{i}{2k}e^{ik(x-s)}, & -\infty<x<s\\ \dfrac{i}{2k}e^{ik(s-x)}, & s<x<\infty\end{cases}$$

System A

$$G(x,s)=\begin{cases}\dfrac{-i}{2k}e^{ik(x+s)}, & -\infty<x<s\\ \dfrac{-i}{2k}e^{-ik(s+x)}, & s<x<\infty\end{cases}$$

System B

$$G(x,s)=\begin{cases}\dfrac{i}{2k}e^{ik(x-s)}-\dfrac{i}{2k}e^{-ik(s+x)}=-\dfrac{1}{k}e^{-iks}\sin kx, & 0<x<s\\ \dfrac{i}{2k}e^{ik(s-x)}-\dfrac{i}{2k}e^{-ik(s+x)}=-\dfrac{1}{k}e^{-ikx}\sin ks, & s<x<\infty\end{cases}$$

(b) Wronskian Approach

$$G(x,s)=\begin{cases}c_1\cos kx, & 0<x<s\\ c_2e^{-ikx}, & s<x<\infty\end{cases}$$

Displacement Continuity at $x=s$

$$c_1\cos ks=c_2e^{-iks}$$

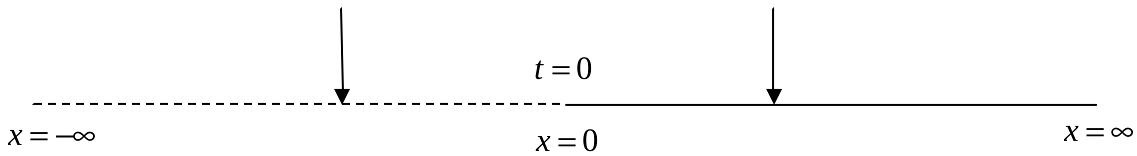
The difference in slope is 1 $x = s$

$$-ikc_2 e^{-iks} + kc_1 \sin ks = 1$$

$$\therefore c_1 = \frac{i}{k} e^{-iks}, \quad \therefore c_2 = \frac{i}{k} \cos ks$$

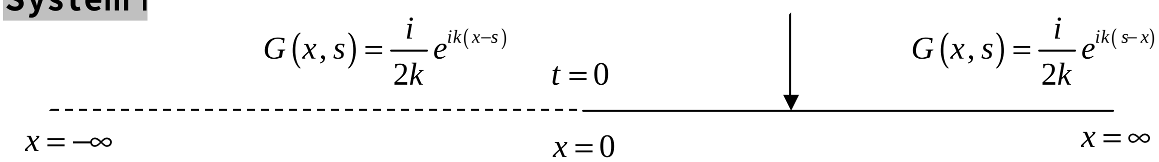
$$G(x,s) = \begin{cases} \frac{i}{k} e^{-iks} \cos kx, & 0 < x < s \\ \frac{i}{k} e^{-ikx} \cos ks, & s < x < \infty \end{cases}$$

(b) Image Method



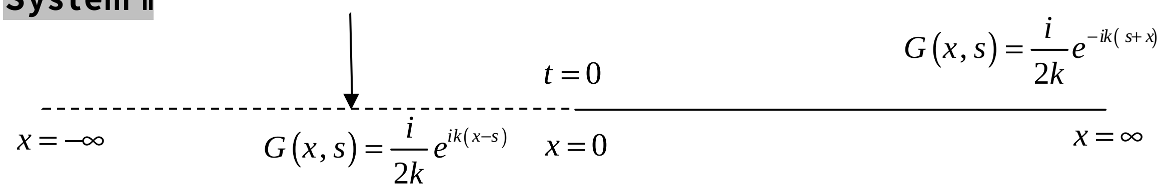
$$=$$

System I



$$+$$

System II



$$G(x,s) = \begin{cases} \frac{i}{2k} e^{ik(x-s)}, & -\infty < x < s \\ \frac{i}{2k} e^{ik(s-x)}, & s < x < \infty \end{cases}$$ **System I**

$$G(x,s) = \begin{cases} \frac{i}{2k} e^{ik(s+x)}, & -\infty < x < s \\ \frac{i}{2k} e^{-ik(s+x)}, & s < x < \infty \end{cases}$$ **System II**

$$G(x,s)=\begin{cases}\dfrac{i}{2k}e^{ik(x-s)}+\dfrac{i}{2k}e^{-ik(s+x)}=\dfrac{i}{k}e^{-iks}\cos kx, & 0<x<s\\ \dfrac{i}{2k}e^{ik(s-x)}+\dfrac{i}{2k}e^{-ik(s+x)}=\dfrac{i}{k}e^{-ikx}\cos ks, & s<x<\infty\end{cases}$$

4.

$R=1$, $r=p$, $q=q$, $f=0$

$$\frac{p\sin q}{1+p^2-2p\cos q}=\begin{cases}\sum_{m=1}^{\infty} p^m \sin mq, & |p|<1\\ \sum_{m=1}^{\infty}\left(\frac{1}{p}\right)^m \sin mq, & |p|>1\end{cases}\rightarrow (a)$$

$$\frac{1-p\cos q}{1+p^2-2p\cos q}=\begin{cases}1+\sum_{m=1}^{\infty} p^m \cos mq, & |p|<1\\ -\sum_{m=1}^{\infty}\left(\frac{1}{p}\right)^m \cos mq, & |p|>1\end{cases}\rightarrow (b)$$

$$\frac{p-\cos q}{1+p^2-2p\cos q}=\begin{cases}-\sum_{m=1}^{\infty} p^{m-1} \cos mq, & |p|<1\\ \frac{1}{p}+\sum_{m=1}^{\infty}\left(\frac{1}{p}\right)^{m+1} \cos mq, & |p|>1\end{cases}\rightarrow (c)$$

$$\frac{(c)\times p-(b)}{p^2-1}\rightarrow \frac{1}{1+p^2-2p\cos q}=\begin{cases}\frac{1}{1-p^2}-2\sum_{m=1}^{\infty} p^m \cos mq, & |p|<1\\ \frac{1}{p^2-1}+2\sum_{m=1}^{\infty}\left(\frac{1}{p}\right)^m \cos mq, & |p|>1\end{cases}\rightarrow (d)$$

$$\frac{(b)\times p-(c)}{1-p^2}\rightarrow \frac{\cos q}{1+p^2-2p\cos q}=\begin{cases}\dfrac{p+\sum_{m=1}^{\infty}\left((p)^{m+1}+(p)^{m-1}\right)\cos mq}{1-p^2}, & |p|<1\\ \dfrac{-\frac{1}{p}-\sum_{m=1}^{\infty}\left(\left(\frac{1}{p}\right)^{m+1}+\left(\frac{1}{p}\right)^{m-1}\right)\cos mq}{1-p^2}, & |p|>1\end{cases}\rightarrow (e)$$

$$\frac{(a)}{p}\rightarrow \frac{\sin q}{1+p^2-2p\cos q}=\begin{cases}\sum_{m=1}^{\infty} p^{m-1} \sin mq, & |p|<1\\ \sum_{m=1}^{\infty}\left(\frac{1}{p}\right)^{m+1} \sin mq, & |p|>1\end{cases}\rightarrow (f)$$

(a)

$$\oint \frac{1}{1+p^2-2p\cos q}dq = \begin{cases} \oint\left(\frac{1}{1-p^2}-2\sum_{m=1}^{\infty} p^m \cos mq\right)dq = \frac{2p}{1-p^2}, & |p|<1 \\ \oint\left(\frac{1}{p^2-1}+2\sum_{m=1}^{\infty}\left(\frac{1}{p}\right)^m \cos mq\right)dq = \frac{2p}{p^2-1}, & |p|>1 \end{cases}$$

(b)

$(e)\times\cos q-(f)\times\sin q \rightarrow$

$$\oint \frac{\cos 2q}{1+p^2-2p\cos q}dq$$

$$= \begin{cases} \oint \frac{p\cos q+\sum_{m=1}^{\infty}\left((p)^{m+1}+(p)^{m-1}\right)\frac{1}{2}\left(\cos(1-m)q+\cos(1+m)q\right)}{1-p^2}, & |p|<1 \\ -\sum_{m=1}^{\infty} p^{m-1}\frac{1}{2}\left(\cos(1-m)q-\cos(1+m)q\right)dq = \frac{2p\,p^2}{1-p^2} & \\ \oint \frac{-\frac{\cos q}{p}-\sum_{m=1}^{\infty}\left(\left(\frac{1}{p}\right)^{m+1}+\left(\frac{1}{p}\right)^{m-1}\right)\frac{1}{2}\left(\cos(1-m)q+\cos(1+m)q\right)}{1-p^2}, & |p|>1 \\ -\sum_{m=1}^{\infty}\left(\frac{1}{p}\right)^{m+1}\frac{1}{2}\left(\cos(1-m)q-\cos(1+m)q\right)dq = \frac{2p}{p^4-p^2} & \end{cases}$$

(c)

$$\oint T(s,x)\,dq = \oint \frac{1-p\cos q}{1+p^2-2p\cos q}dq$$

$$= \begin{cases} \oint 1+\sum_{m=1}^{\infty} p^m \cos mq\,dq = 2p, & |p|<1 \\ \oint -\sum_{m=1}^{\infty}\left(\frac{1}{p}\right)^m \cos mq\,dq = 0, & |p|>1 \end{cases}$$

(d)

$$\oint \frac{\sin q}{1+p^2-2p\cos q}dq = \begin{cases} \oint \sum_{m=1}^{\infty}(p)^{m-1}\sin(mq)dq = 0, & |p|<1 \\ \oint \sum_{m=1}^{\infty}\left(\frac{1}{p}\right)^{m+1}\sin(mq)\,dq = 0, & |p|>1 \end{cases}$$

(e)

$$\oint \ln\sqrt{\left(1+p^2-2p\cos q\right)}dq = \oint \ln r\,dq = \begin{cases} \oint -\sum_{m=1}^{\infty}\frac{1}{m}p^m \cos mq\,dq = 0, & |p|<1 \\ \oint \ln p-\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{1}{p}\right)^m \cos mq\,dq = 2p\ln p, & |p|>1 \end{cases}$$

5.

$$M(s,x)=\begin{cases}\nabla_s L(s,x)\bullet n_s=\left(\dfrac{\partial L(s,x)}{\partial s_1}\underset{\sim}{i}+\dfrac{\partial L(s,x)}{\partial s_2}\underset{\sim}{j}\right)\bullet\left(n_1\underset{\sim}{i}+n_2\underset{\sim}{j}\right)\\ \nabla_x T(s,x)\bullet n_x=\left(\dfrac{\partial T(s,x)}{\partial x_1}\underset{\sim}{i}+\dfrac{\partial T(s,x)}{\partial x_2}\underset{\sim}{j}\right)\bullet\left(\overline{n_1}\underset{\sim}{i}+\overline{n_2}\underset{\sim}{j}\right)\end{cases}$$

$$L(s,x)=\frac{-(s_1-x_1)}{r^2}\overline{n_1}+\frac{-(s_2-x_2)}{r^2}\overline{n_2}\text{,}\quad T(s,x)=\frac{(s_1-x_1)}{r^2}n_1+\frac{(s_2-x_2)}{r^2}n_2$$

$$M(s,x)=\begin{cases}\dfrac{2y_1^{\,2}}{r^4}n_1\overline{n_1}+\dfrac{2y_1y_2}{r^4}n_1\overline{n_2}+\dfrac{2y_1y_2}{r^4}\overline{n_1}n_2+\dfrac{2y_2^{\,2}}{r^4}n_2\overline{n_2}-\dfrac{n_1\overline{n_1}+n_2\overline{n_2}}{r^2} & From\quad T(s,x)\\ \dfrac{2y_1^{\,2}}{r^4}n_1\overline{n_1}+\dfrac{2y_1y_2}{r^4}n_1\overline{n_2}+\dfrac{2y_1y_2}{r^4}\overline{n_1}n_2+\dfrac{2y_2^{\,2}}{r^4}n_2\overline{n_2}-\dfrac{n_1\overline{n_1}+n_2\overline{n_2}}{r^2} & From\quad L(s,x)\end{cases}$$

$$M(s,x)=\frac{\partial T(s,x)}{\partial r}=\begin{cases}\displaystyle\sum_{m=1}^{\infty}m\left(\frac{r^{m-1}}{R^{m+1}}\right)(\cos m(q-f)), & R\geq r\\ \displaystyle\sum_{m=1}^{\infty}m\left(\frac{R^{m-1}}{r^{m+1}}\right)(\cos m(q-f)), & R<r\end{cases}$$

$$M(s,x)=\frac{\partial L(s,x)}{\partial R}=\begin{cases}\displaystyle\sum_{m=1}^{\infty}m\left(\frac{r^{m-1}}{R^{m+1}}\right)(\cos m(q-f)), & R\geq r\\ \displaystyle\sum_{m=1}^{\infty}m\left(\frac{R^{m-1}}{r^{m+1}}\right)(\cos m(q-f)), & R<r\end{cases}$$