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1. $\dfrac{d^4 U(x,s)}{dx^4} + q^4 U(x,s) = \delta(x-s),\ -\infty < x < \infty$

(1) Is U(x, s) singular
(2) Is U(x, s) symmetric
(3) Is U(x, s) degenerate form

ANS

$$F\left\{\frac{d^4 U(x,s)}{dx^4} + q^4 U(x,s)\right\} = F\{\delta(x-s)\}$$

$\rightarrow (k^4+q^4)\,u(k,s) = e^{-iks}$

$\rightarrow u(k,s) = \dfrac{e^{-iks}}{(k^4+q^4)}$

→ Inverse Fourier Transform

$\rightarrow U(x,s) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} u(k,s)\, e^{ikx}\, dk = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \dfrac{1}{(k^4+q^4)} e^{ik(x-s)}\, dk$

Let $f(z,s) = \dfrac{e^{iz(x-s)}}{(z^4+q^4)}$

$(z^4+q^4) = 0 \rightarrow z = qe^{i\left(\frac{\pi+2k\pi}{4}\right)}$ $k = 0, 1, 2, 3$ 爲單極點

(a) $x > s$

但只有$z_1 = qe^{i\frac{\pi}{4}}$, $z_2 = qe^{i\frac{3\pi}{4}}$ 在上半平面且其殘值爲

$\text{Res}\, u(z_1) = \text{Limit}\left[\dfrac{e^{ik(x-s)}}{(k - q e^{i\frac{3}{4}\pi})(k - q e^{i\frac{5}{4}\pi})(k - q e^{i\frac{7}{4}\pi})}, k \rightarrow q e^{i\frac{\pi}{4}}\right]$

$= \dfrac{e^{(-1)^{3/4} q (x-s)}}{4q^3 (-1)^{3/4}}$

$\text{Res}\, u(z_2) = \text{Limit}\left[\dfrac{e^{ik(x-s)}}{(k - q e^{i\frac{1}{4}\pi})(k - q e^{i\frac{5}{4}\pi})(k - q e^{i\frac{7}{4}\pi})}, k \rightarrow q e^{i\frac{3\pi}{4}}\right]$

$= \dfrac{e^{(-1)^{1/4} q (s-x)}}{4q^3 (-1)^{1/4}}$

故

$\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} u(k,s)\, e^{iks}\, dk = \dfrac{2\pi i}{2\pi}[\text{Res}\, u(z_1) + \text{Res}\, u(z_2)] = i\left\{-\dfrac{(-1)^{1/4}\left(e^{(-1)^{-\frac{3}{4}} q(s-x)} + i\, e^{(-1)^{\frac{1}{4}} q(s-x)}\right)}{4q^3}\right\}$

$\rightarrow \left\{-\dfrac{(-1)^{1/4}\left(i e^{(-1)^{-3/4} q(s-x)} - e^{(-1)^{1/4} q(s-x)}\right)}{4q^3}\right\}$

特解 + 補解 = 特解

$$\to \text{Limit}\left[\frac{(-1)^{1/4}\left(-i e^{(-1)^{-3/4} q (s-x)} + e^{(-1)^{1/4} q (s-x)}\right)}{4 q^3} - \frac{(-1)^{1/4} - (-1)^{1/4} i}{4 q^3}, q \to 0\right]$$

L ' Hospital rule

$$\to \text{Limit}\left[\frac{(-1)^{1/4}\left((-1)^{1/4} e^{(-1)^{1/4} q (s-x)} (s-x) + (-1)^{1/4} e^{(-1)^{3/4} q (-s+x)} (-s+x)\right)}{12 q^2}, q \to 0\right]$$

$$\to \text{Limit}\left[\frac{(-1)^{1/4}\left(i e^{(-1)^{1/4} q (s-x)} (s-x)^2 - e^{(-1)^{3/4} q (-s+x)} (-s+x)^2\right)}{24 q}, q \to 0\right]$$

$$\to \text{Limit}\left[\frac{1}{24}\left((-1)^{1/4}\left((-1)^{3/4} e^{(-1)^{1/4} q (s-x)} (s-x)^3 - (-1)^{3/4} e^{(-1)^{3/4} q (-s+x)} (-s+x)^3\right)\right), q \to 0\right]$$

$$\to -\frac{1}{12} (s-x)^3$$

$$\to \frac{1}{12} (x-s)^3$$

(b) $x < s$

$$U(x, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(k, s) e^{i k x} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(k^4 + q^4)} e^{i k (x-s)} dk$$

Let $f(z, s) = \dfrac{e^{i z (x-s)}}{(z^4 + q^4)}$

但只有$z_3 = q e^{i \frac{5\pi}{4}}$, $z_4 = q e^{i \frac{7\pi}{4}}$ 在下半平面且其殘值爲

$$\text{Res}\, u(z_3) = \text{Limit}\left[\frac{e^{i k (x-s)}}{\left(k - q e^{i \frac{1}{4}\pi}\right)\left(k - q e^{i \frac{3}{4}\pi}\right)\left(k - q e^{i \frac{7}{4}\pi}\right)}, k \to q e^{i \frac{5\pi}{4}}\right]$$

$$= \frac{\left(e^{(-1)^{3/4} q (s-x)}\right)}{-4 (-1)^{3/4} q^3}$$

$$\text{Res}\, u(z_4) = \text{Limit}\left[\frac{e^{i k (x-s)}}{\left(k - q e^{i \frac{1}{4}\pi}\right)\left(k - q e^{i \frac{3}{4}\pi}\right)\left(k - q e^{i \frac{5}{4}\pi}\right)}, k \to q e^{i \frac{7\pi}{4}}\right]$$

$$= \frac{e^{(-1)^{1/4} q (-s+x)}}{-4 (-1)^{1/4} q^3}$$

故

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} u(k, s) e^{i k s} dk = \frac{-2\pi i}{2\pi} [\text{Res}\, u(z_3) + \text{Res}\, u(z_4)] = -i\left\{\frac{e^{-(-1)^{3/4} q (x-s)}}{-4 q^3 (-1)^{3/4}} + \frac{e^{(-1)^{1/4} q (s-x)}}{-4 q^3 (-1)^{1/4}}\right\}$$

$$\to -\frac{(-1)^{3/4}\left(e^{(-1)^{3/4} q (s-x)} + i e^{(-1)^{1/4} q (-s+x)}\right)}{4 q^3}$$

特解 + 補解 = 特解

$$\to \left\{-\frac{(-1)^{3/4}\left(e^{(-1)^{\frac{1}{4}} q (s-x)} - i e^{(-1)^{3/4} q (s-x)}\right)}{4 q^3} + \frac{(-1)^{3/4} - (-1)^{3/4} i}{4 q^3}, q \to 0\right\}$$

L ' Hospital rule

$$\to \text{Limit}\left[\frac{1}{12 q^2}\left(-(-1)^{3/4}\left((-1)^{3/4} e^{(-1)^{3/4} q (s-x)} (s-x) + (-1)^{3/4} e^{(-1)^{1/4} q (-s+x)} (-s+x)\right)\right), q \to 0\right]$$

$$\to \text{Limit}\left[\frac{-(-1)^{3/4}\left(-i\, e^{(-1)^{3/4} q (s-x)} (s-x)^2 - e^{(-1)^{1/4} q (-s+x)} (-s+x)^2\right)}{24 q}, q \to 0\right]$$

$$\to \text{Limit}\left[\frac{1}{24}\left(-(-1)^{3/4}\left((-1)^{1/4} e^{(-1)^{3/4} q (s-x)} (s-x)^3 - (-1)^{1/4} e^{(-1)^{1/4} q (-s+x)} (-s+x)^3\right)\right), q \to 0\right]$$

$$\to \frac{1}{12} (s-x)^3$$

$$U(x, s)\begin{cases} \frac{1}{12} (x-s)^3 & x > s \\ \frac{1}{12} (s-x)^3 & x < s \end{cases}$$

(1) **regular**
(2) **symmetric**
(3) **degenerate form**

2. $\dfrac{d^2 G(x, s)}{d x^2} = \delta(x - s),\ 0 < x < L$

Subjected to G(0, s) = 0, G'(l, s) = 0

(1) Is U(x, s) singular

(2) Is U(x, s) symmetric

(3) Is U(x, s) degenerate form

ANS

as $0 < x < s$

$$\frac{d^2 g_1(x)}{d x^2} = 0 \qquad 0 < x < s$$

$\to g_1(x) = a + bx$ B.C. $G(0, s) = 0$

$\to g_1(x) = bx$

as $s < x < l$

$$\frac{d^2 g_2(x)}{d x^2} = 0 \qquad s < x < l$$

$\to g_2(x) = c + dx$ B.C. $G'(l, s) = 0$

$\to g_2(x) = c$

Satisfied two conditions

$g_1(s) = g_2(s)$

$g_2'(s^+) - g_1'(s^-) = 1$

$$\to \begin{cases} b = -1 \\ c = -s \end{cases}$$

The Greens Function is

$$G(x, s) = \begin{cases} -x, & 0 < x < s \\ -s, & s < x < l \end{cases}$$

(1) regular
(2) symmetric
(3) degenerate form

Null