Given a casual function $f(t) = e^{-\xi t}\cos[t]$, $t > 0$, otherwise $f(t) = 0$

(1) Please find $f_e(t)$ and $f_o(t)$

(2) Plot $f_e(t)$ and $f_o(t)$

(3) Please find its Fourier transform

(4) Check its Hilbert transform pair using complex inegrals

(5) By taking the limit of $\xi \to 0$, recheck the results of the previous homeworks.

Hint : $\text{Lim}\, S_k(x) = \lim_{k\to 0} \frac{1}{\pi} \frac{k}{(1+k^2 x^2)} = \delta(x)$

Sol :

(1)

$$f(t) = \begin{cases} e^{-\xi t}\cos[t] & t > 0 \\ 0 & t \le 0 \end{cases}$$

$$f(-t) = \begin{cases} 0 & t > 0 \\ e^{\xi t}\cos[t] & t \le 0 \end{cases}$$

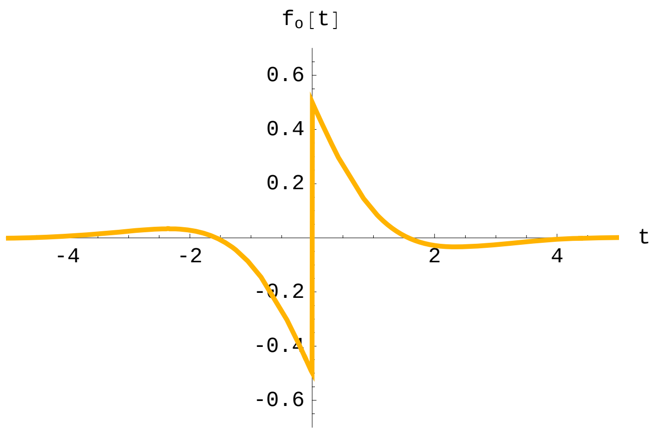
$$f_e = \frac{1}{2}(f(t) + f(-t)) = \begin{cases} \frac{1}{2} e^{-\xi t}\cos[t] & t > 0 \\ \frac{1}{2} e^{\xi t}\cos[t] & t \le 0 \end{cases}$$

$$f_o = \frac{1}{2}(f(t) + f(-t)) = \begin{cases} \frac{1}{2} e^{-\xi t}\cos[t] & t > 0 \\ \frac{-1}{2} e^{\xi t}\cos[t] & t \le 0 \end{cases}$$

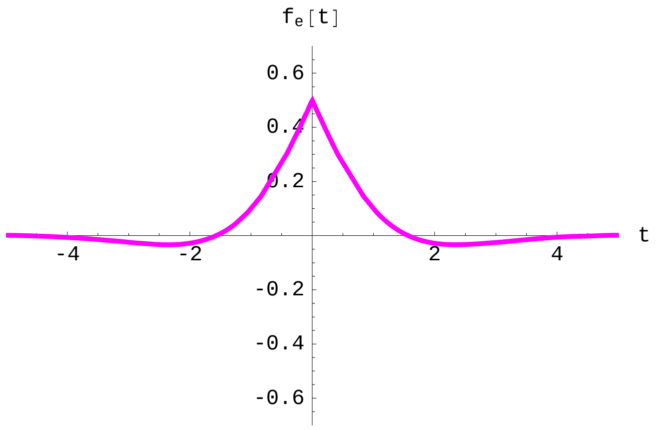
(2)

$\xi = 1$

$f_o[t]$

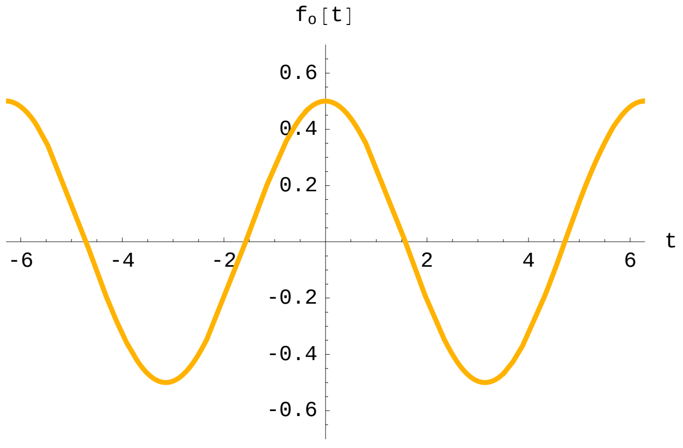


$f_e[t]$

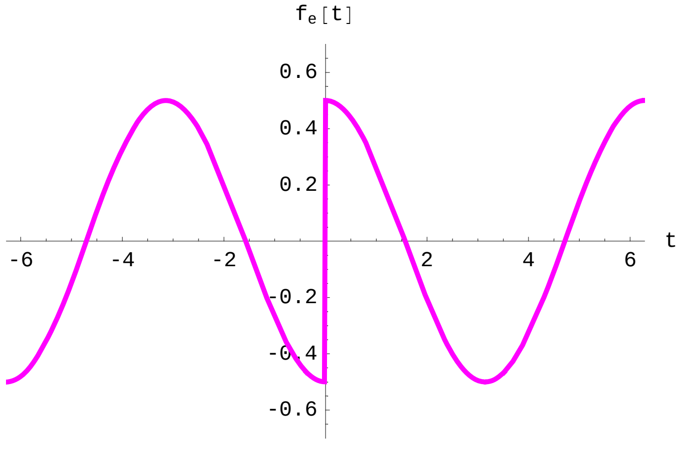


ξ = 0

f_o[t]



f_e[t]



(3)

$$F_e\ (\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_e\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{1}{2} e^{\xi t} \cos[t]\, e^{-i\omega t}\, e\, dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{2} e^{-\xi t} \cos[t]\, e^{-i\omega t}\, e\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{1}{4} e^{\xi t} (e^{it} + e^{-it})\, e^{-i\omega t}\, e\, dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{4} e^{-\xi t} (e^{it} + e^{-it})\, e^{-i\omega t}\, e\, dt$$

$$= \frac{1}{4\sqrt{2\pi}} \int_{-\infty}^{0} e^{(\xi + i - i\omega) t} + e^{(\xi - i - i\omega) t}\, dt + \frac{1}{4\sqrt{2\pi}} \int_{0}^{\infty} e^{(-\xi + i - i\omega) t} + e^{(-\xi - i - i\omega) t}\, e\, dt$$

$$= \frac{1}{4\sqrt{2\pi}} \left(\frac{1}{\xi + i(1-\omega)} + \frac{1}{\xi - i(1+\omega)} \right) + \frac{1}{4\sqrt{2\pi}} \left(\frac{-1}{-\xi + i(1-\omega)} + \frac{-1}{-\xi - i(1+\omega)} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\xi^3 + \xi(1+\omega^2)}{(\xi^2 + 1 - \omega^2)^2 + 4\xi^2\omega^2}$$

$$-i F_o(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_o\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{-1}{2} e^{\xi t} \cos[t]\, e^{-i\omega t}\, e\, dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{2} e^{-\xi t} \cos[t]\, e^{-i\omega t}\, e\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{-1}{4} e^{\xi t} (e^{it} + e^{-it})\, e^{-i\omega t}\, e\, dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{4} e^{-\xi t} (e^{it} + e^{-it})\, e^{-i\omega t}\, e\, dt$$

$$= \frac{-1}{4\sqrt{2\pi}} \int_{-\infty}^{0} e^{(\xi + i - i\omega) t} + e^{(\xi - i - i\omega) t}\, dt + \frac{1}{4\sqrt{2\pi}} \int_{0}^{\infty} e^{(-\xi + i - i\omega) t} + e^{(-\xi - i - i\omega) t}\, e\, dt$$

$$= \frac{-1}{4\sqrt{2\pi}} \left(\frac{1}{\xi + i(1-\omega)} + \frac{1}{\xi - i(1+\omega)} \right) + \frac{1}{4\sqrt{2\pi}} \left(\frac{-1}{-\xi + i(1-\omega)} + \frac{-1}{-\xi - i(1+\omega)} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{-i\omega(\xi^2 + \omega^2 - 1)}{(\xi^2 + 1 - \omega^2)^2 + 4\xi^2\omega^2}$$

(4)

$$\frac{1}{\sqrt{2\pi}} F_e(\omega) * \sqrt{\frac{2}{\pi}} \left(\frac{-i}{\omega} \right)$$

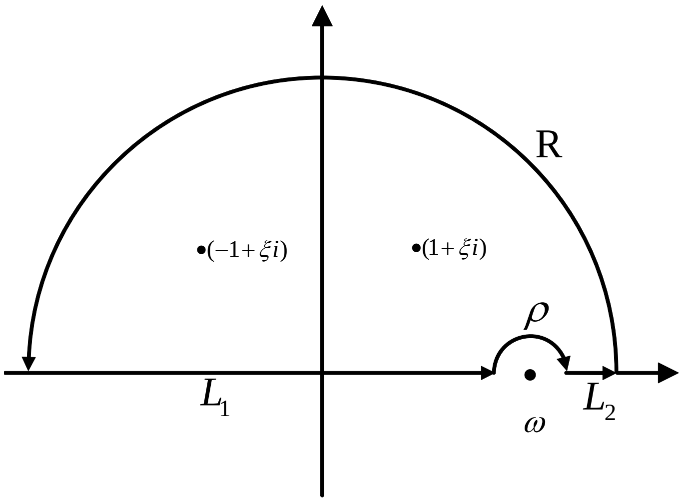
$$= \frac{-i}{\pi\sqrt{2\pi}} \left(\frac{\xi^3 + \xi(1+\omega^2)}{(\xi^2 + 1 - \omega^2)^2 + 4\xi^2\omega^2} * \frac{1}{\omega} \right)$$

$$= \frac{-i}{\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\xi^3 + \xi(1+\tau^2)}{((\xi^2 + 1 - \tau^2)^2 + 4\xi^2\tau^2)} \frac{1}{\omega - \tau}\, d\tau$$

$$\int_{-\infty}^{\infty} \frac{\xi^3 + \xi(1+\tau^2)}{((\xi^2 + 1 - \tau^2)^2 + 4\xi^2\tau^2)} \frac{1}{\omega - \tau}\, d\tau$$

四個一階**poles**位於

$\tau = \pm 1 + i\xi\ ,\ \pm 1 - i\xi\ ,\ \omega$



$$\text{Res}[1+\xi\, i] = \frac{1}{\xi + i\,(\omega - 1)}$$

$$\text{Res}[-1+\xi\, i] = \frac{1}{\xi + i\,(\omega + 1)}$$

$$\text{Res}[\omega] = -\frac{\xi^3 + \xi\,(1+\omega^2)}{((\xi^2 + 1 - \omega^2)^2 + 4\,\xi^2\,\omega^2)}$$

$$\int_{-\infty}^{\infty} \frac{\xi^3 + \xi\,(1+\tau^2)}{((\xi^2 + 1 - \tau^2)^2 + 4\,\xi^2\,\tau^2)}\ \frac{1}{\omega - \tau}\, d\tau = 2\,\pi\, i\,[\text{Res}[1+\xi\, i] + \text{Res}[-1+\xi\, i]]$$

$$= \int_{L_1+L_2} + \int_{\rho+R} \left(\frac{\xi^3 + \xi\,(1+\tau^2)}{((\xi^2 + 1 - \tau^2)^2 + 4\,\xi^2\,\tau^2)}\ \frac{1}{\omega - \tau}\right) d\tau$$

$$\Longrightarrow \int_{L_1+L_2} \left(\frac{\xi^3 + \xi\,(1+\tau^2)}{((\xi^2 + 1 - \tau^2)^2 + 4\,\xi^2\,\tau^2)}\ \frac{1}{\omega - \tau}\right) d\tau = 2\,\pi\, i\,[\text{Res}[1+\xi\, i] + \text{Res}[-1+\xi\, i]] + \pi i\,[\text{Res}[\omega]]$$

$$= 2\,\pi\, i\,\left(\frac{1}{\xi + i\,(\omega - 1)} + \frac{1}{\xi + i\,(\omega + 1)}\right) + \pi i\,\left(-\frac{\xi^3 + \xi\,(1+\omega^2)}{((\xi^2 + 1 - \omega^2)^2 + 4\,\xi^2\,\omega^2)}\right)$$

$$= \frac{\pi\,\omega\,(-1 + \xi^2 + \omega^2)}{\xi^4 + (-1 + \omega^2)^2 + 2\,\xi^2\,(1 + \omega^2)}$$

$$\frac{1}{\sqrt{2\,\pi}}\ F_e\,(\omega) * \sqrt{\frac{2}{\pi}}\ \left(\frac{-i}{\omega}\right)$$

$$= \frac{-i}{\pi\,\sqrt{2\,\pi}}\ \int_{-\infty}^{\infty} \frac{\xi^3 + \xi\,(1+\tau^2)}{((\xi^2 + 1 - \tau^2)^2 + 4\,\xi^2\,\tau^2)}\ \frac{1}{\omega - \tau}\, d\tau$$

$$= \frac{1}{\sqrt{2\,\pi}}\ \frac{-i\,\omega\,(-1 + \xi^2 + \omega^2)}{\xi^4 + (-1 + \omega^2)^2 + 2\,\xi^2\,(1 + \omega^2)}$$

$$\frac{1}{\sqrt{2\,\pi}}\ F_o\,(\omega) * \sqrt{\frac{2}{\pi}}\ \left(\frac{-i}{\omega}\right)$$

$$= \frac{1}{\pi\,\sqrt{2\,\pi}}\ \left(\frac{\omega\,(\xi^2 + \omega^2 - 1)}{(\xi^2 + 1 - \omega^2)^2 + 4\,\xi^2\,\omega^2} * \frac{1}{\omega}\right)$$

$$= \frac{1}{\pi\,\sqrt{2\,\pi}}\ \int_{-\infty}^{\infty} \frac{\tau\,(\xi^2 + \tau^2 - 1)}{(\xi^2 + 1 - \tau^2)^2 + 4\,\xi^2\,\tau^2} * \frac{1}{\omega - \tau}\, d\tau$$

$$\int_{-\infty}^{\infty} \frac{\tau\,(\xi^2 + \tau^2 - 1)}{(\xi^2 + 1 - \tau^2)^2 + 4\,\xi^2\,\tau^2} * \frac{1}{\omega - \tau}\, d\tau$$

$$\text{Res}[1+\xi\, i] = \frac{1}{4\,(-1 - i\,\xi + \omega)}$$

$$\text{Res}[-1+\xi\, i] = \frac{1}{4\,(1 - i\,\xi + \omega)}$$

$$\text{Res}[\omega] = -\frac{\omega\,(\xi^2 + \omega^2 - 1)}{(\xi^2 + 1 - \omega^2)^2 + 4\,\xi^2\,\omega^2}$$

$$\int_{-\infty}^{\infty} \frac{\tau\,(\xi^2 + \tau^2 - 1)}{(\xi^2 + 1 - \tau^2)^2 + 4\,\xi^2\,\tau^2} * \frac{1}{\omega - \tau}\, d\tau = 2\,\pi\, i\,[\text{Res}[1+\xi\, i] + \text{Res}[-1+\xi\, i]] + \pi i\,[\text{Res}[\omega]]$$

$$= -\frac{\pi\,(\xi^3 + \xi\,(1 + \omega^2))}{(\xi^2 + 1 - \omega^2)^2 + 4\,\xi^2\,\omega^2}$$

$$\frac{1}{\sqrt{2\,\pi}}\, F_o\,(\omega) * \sqrt{\frac{2}{\pi}}\, \left(\frac{-i}{\omega}\right)$$

$$= \frac{1}{\pi\,\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \frac{\tau\,(\xi^2 + \tau^2 - 1)}{(\xi^2 + 1 - \tau^2)^2 + 4\,\xi^2\,\tau^2} * \frac{1}{\omega - \tau}\, d\tau$$

$$= \frac{-1}{\sqrt{2\,\pi}}\ \frac{\xi^3 + \xi\,(1 + \omega^2)}{(\xi^2 + 1 - \omega^2)^2 + 4\,\xi^2\,\omega^2}$$

(5)

$$\text{Lim}[F_e\,(\omega)] = \text{Lim}_{\xi\to 0}\left[\frac{1}{\sqrt{2\,\pi}}\ \frac{\xi^3 + \xi\,(1 + \omega^2)}{(\xi^2 + 1 - \omega^2)^2 + 4\,\xi^2\,\omega^2}\right]$$

$$= \frac{1}{\sqrt{2\,\pi}}\ \text{Lim}_{\xi\to 0}\left[\frac{\xi^3 + \xi\,(1 + \omega^2)}{(\omega + 1 + \xi\, i)\,(\omega + 1 - \xi\, i)\,(\omega - 1 + \xi\, i)\,(\omega - 1 - \xi\, i)}\right]$$

$$= \frac{1}{\sqrt{2\,\pi}}\ \text{Lim}_{\xi\to 0}\left[\frac{\xi^3 + \xi\,(1 + \omega^2)}{(\xi^2 + (1 + \omega)^2)\,(\xi^2 + (-1 + \omega)^2)}\right]$$

$$= \frac{1}{\sqrt{2\,\pi}}\ \text{Lim}_{\xi\to 0}\left[\frac{1}{2}\left(\frac{\xi}{(\xi^2 + (1 + \omega)^2)} + \frac{\xi}{(\xi^2 + (-1 + \omega)^2)}\right)\right]$$

$$= \frac{\pi}{2\,\sqrt{2\,\pi}}\ \text{Lim}_{\xi\to 0}\left[\frac{1}{\pi}\left(\frac{\xi}{(\xi^2 + (1 + \omega)^2)} + \frac{\xi}{(\xi^2 + (-1 + \omega)^2)}\right)\right]$$

$$= \frac{\sqrt{2\,\pi}}{4}\ (\delta\,(\omega + 1) + \delta\,(\omega - 1))$$