工程數學（三）—複變　期末考

8:20-10:10, Jan. 16, 1995

I. By using the following residue theory and Parseval's equality,

$$\int_C G(\bar{\omega})d\bar{\omega} = 2\pi i \sum Res, \quad \int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\bar{\omega})F^*(\bar{\omega})d\bar{\omega}$$

Find

$$\int_{-\infty}^{\infty} F(\bar{\omega})F^*(\bar{\omega})d\bar{\omega} = ?, \quad where \quad F(\bar{\omega}) = \frac{1}{2i\xi\omega\bar{\omega} + \omega^2} \quad (\xi,\ \omega\ are\ constants)$$

(1). Frequency domain approach by residue. (10%)

(2). Time domain approach by Parseval's equality. (10%)

(3). If the solutions of (1) and (2) are the same. Any physical meaning ? (10%)

(4). Explain why the solution is infinite as $\xi = 0$ or $\omega = 0$?(10%)

(5). If $F(\bar{\omega}) = \frac{1}{-\bar{\omega}^2 + 2i\xi\omega\bar{\omega} + \omega^2}$, repeat (1) (10%) and (2) (10%). $(0 < \xi < 1)$

II. If $0 <| z - 2 |< 2$, express $\frac{1}{(z-2)(z-4)}$ in terms of Laurent Series.(10%)

III. If $x(t)$ satisfies the casual effect, then the real and imaginary parts of $\bar{X}(\bar{\omega})$ satisfy Hilbert transform

$$\bar{X}(\bar{\omega}) = \frac{1}{(-\bar{\omega}^2 + 2i\xi\omega\bar{\omega} + \omega^2)} = \bar{X}_R(\bar{\omega}) + \bar{X}_I(\bar{\omega})i$$

$$\bar{X}_R(\bar{\omega}) = \frac{\omega^2 - \bar{\omega}^2}{(\omega^2 - \bar{\omega}^2)^2 + 4\xi^2\omega^2\bar{\omega}^2}, \quad \bar{X}_I(\bar{\omega}) = \frac{-2\xi\omega\bar{\omega}}{(\omega^2 - \bar{\omega}^2)^2 + 4\xi^2\omega^2\bar{\omega}^2}$$

Proof of Hilbert transform pair using theory of residue (20%):

$$-\bar{X}_I(\bar{\omega}) = \int_{-\infty}^{\infty} \frac{\bar{X}_R(u)}{\pi(\bar{\omega} - u)}du, \quad \bar{X}_R(\bar{\omega}) = \int_{-\infty}^{\infty} \frac{\bar{X}_I(u)}{\pi(\bar{\omega} - u)}du$$

(Hint: prove both cases together by introducing i for the second case)

IV. Given the real part and imaginary parts of

$$\begin{aligned} \bar{X}(\bar{\omega}) &= \frac{1}{-\bar{\omega}^2 + \omega^2(1 \pm i\eta)}; (+, \bar{\omega} > 0; -, \bar{\omega} < 0) \\ &= \frac{\omega^2 - \bar{\omega}^2}{(\omega^2 - \bar{\omega}^2)^2 + (\omega^2\eta)^2} \mp \frac{\omega^2\eta}{(\omega^2 - \bar{\omega}^2)^2 + (\omega^2\eta)^2} i \end{aligned} \tag{1}$$

$$\bar{X}_R(\bar{\omega}) = \frac{\omega^2 - \bar{\omega}^2}{(\omega^2 - \bar{\omega}^2)^2 + (\omega^2\eta)^2}, \quad \bar{X}_I(\bar{\omega}) = \mp \frac{\omega^2\eta}{(\omega^2 - \bar{\omega}^2)^2 + (\omega^2\eta)^2}$$

Prove that the real and imaginary parts do not satisfy the Hilbert transform pair.(20%)

$$-\bar{X}_I(\bar{\omega}) \neq \int_{-\infty}^{\infty} \frac{\bar{X}_R(u)}{\pi(\bar{\omega} - u)}du, \quad \bar{X}_R(\bar{\omega}) \neq \int_{-\infty}^{\infty} \frac{\bar{X}_I(u)}{\pi(\bar{\omega} - u)}du$$