Given a casual function f (t) = cos[t] , t > 0, otherwise f (t) = 0
(1) Please find f_e (t) and f_o (t)
(2) Plot f_e (t) and f_o (t)
(3) Please find its Fourier transform
(4) Check its Hilbert transform pair using complex inegrals
Sol :
(1)

$$f(t) = \begin{cases} \cos[t] & t > 0 \\ 0 & t \le 0 \end{cases}$$

$$f(-t) = \begin{cases} 0 & t > 0 \\ \cos[t] & t \le 0 \end{cases}$$

$$f_e = \frac{1}{2}(f(t) + f(-t)) = \begin{cases} \frac{1}{2}\cos[t] & t > 0 \\ \frac{1}{2}\cos[t] & t \le 0 \end{cases}$$

$$f_o = \frac{1}{2}(f(t) + f(-t)) = \begin{cases} \frac{1}{2}\cos[t] & t > 0 \\ \frac{-1}{2}\cos[t] & t \le 0 \end{cases}$$

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z1[t_] := If[t > 0, Cos[t]/2, Cos[t]/2]
z2[t_] := If[t > 0, Cos[t]/2, -Cos[t]/2]
z3[t_] := If[t > 0, Cos[t], 0]
g1 = Plot[z1[t], {t, -3 π, 3 π}, PlotStyle → {RGBColor[1, 0.7, 0], Thickness[0.008]},
 AxesLabel → {"t", "fo[t]"}, PlotRange → {{2 π, -2 π}, {-0.7, 0.7}}]
g2 = Plot[z2[t], {t, -3 π, 3 π}, PlotStyle → {RGBColor[1, 0, 1], Thickness[0.008]},
 AxesLabel → {"t", "fe[t]"}, PlotRange → {{2 π, -2 π}, {-0.7, 0.7}}]
g3 = Plot[z3[t], {t, -3 π, 3 π}, PlotStyle → {RGBColor[0, 1, 1], Thickness[0.008]},
 AxesLabel → {"t", "f[t]"}, PlotRange → {{2 π, -2 π}, {-1.3, 1.3}}]
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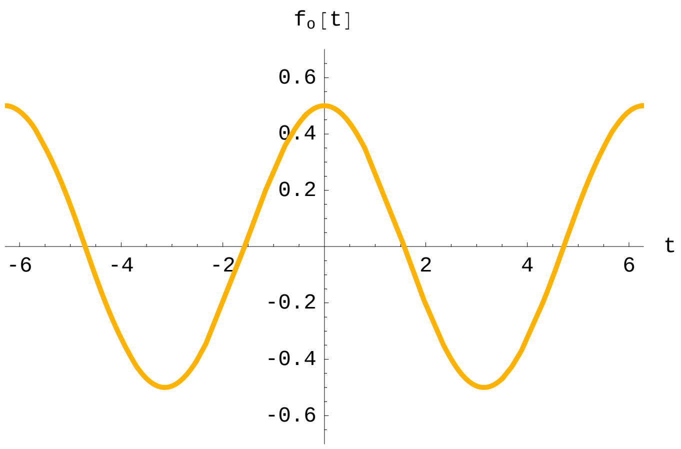

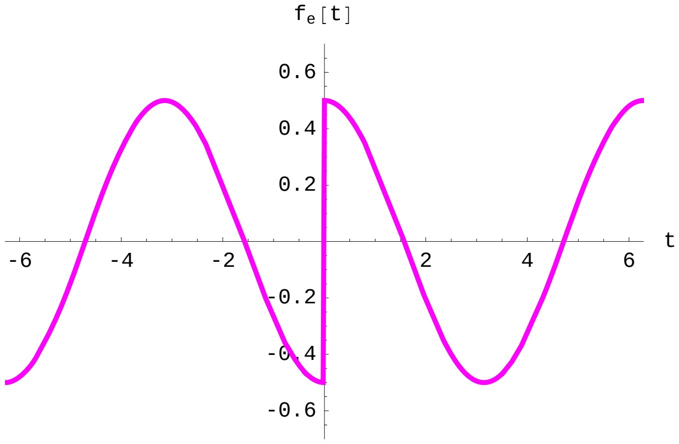

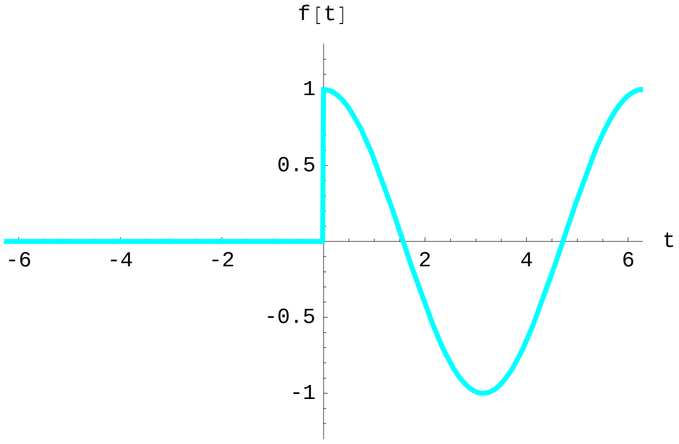

(3)

$$F_e(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f_e * e^{-i\omega t})\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{2} \cos[t] * e^{-i\omega t}\right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{2} \left(\frac{1}{2} (e^{it} + e^{-it})\right) * e^{-i\omega t}\right) dt$$

$$= \frac{1}{4\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{i(1-\omega)t} + e^{-i(1+\omega)t})\, dt$$

$$= \frac{1}{4\sqrt{2\pi}} * 2\pi (\delta(\omega - 1) + \delta(\omega + 1))$$

$$= \frac{\sqrt{2\pi}}{4} (\delta(\omega - 1) + \delta(\omega + 1))$$

$$-i F_o(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f_o * e^{-i\omega t})\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} \left(\frac{-1}{2} \cos[t] * e^{-i\omega t}\right) dt + \int_{0}^{\infty} \left(\frac{1}{2} \cos[t] * e^{-i\omega t}\right) dt\right)$$

$$= \frac{-1}{4\sqrt{2\pi}} \left(\int_{-\infty}^{0} (e^{i(1-\omega)t} + e^{-i(1+\omega)t})\, dt - \int_{0}^{\infty} (e^{i(1-\omega)t} + e^{-i(1+\omega)t})\, dt\right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{i\omega}{1 - \omega^2}$$

(4)

$$\frac{1}{\sqrt{2\pi}} F_e(\omega) * \sqrt{\frac{2}{\pi}} \left(\frac{-i}{\omega}\right) = \int_{-\infty}^{\infty} \left(\frac{\sqrt{2\pi}}{4} (\delta(\tau-1) + \delta(\tau+1))\right) \sqrt{\frac{2}{\pi}} \left(\frac{-i}{\omega-\tau}\right) d\tau$$

$$= \frac{1}{\sqrt{2\pi}} \frac{-i}{2} \int_{-\infty}^{\infty} (\delta(\tau-1) + \delta(\tau+1)) \left(\frac{1}{\omega-\tau}\right) d\tau$$

$$= \frac{1}{\sqrt{2\pi}} \frac{-i}{2} \left(\frac{1}{\omega-1} + \frac{1}{\omega+1}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{-i}{2} \frac{2\omega}{\omega^2-1}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{i\omega}{1-\omega^2}$$