Given a casual function $f(t)=e^{-t}\cos(t), t>0, otherwise\ f(t)=0$

(1) Please find $f_e(t)$ and $f_o(t)$

(2) Plot $f_e(t)$ and $f_o(t)$

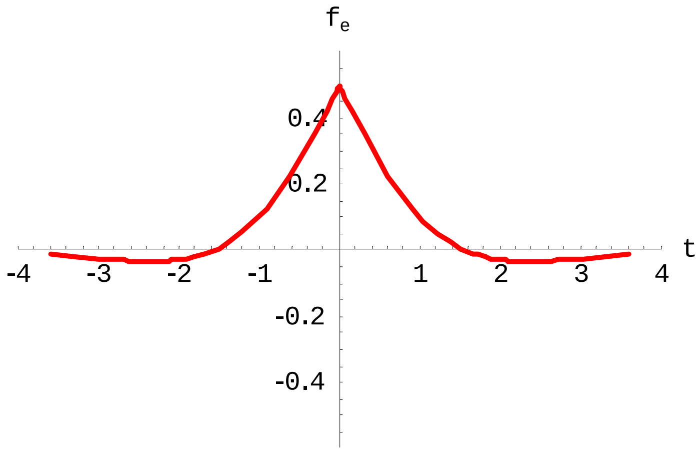
(3) Please find its Fourier transform

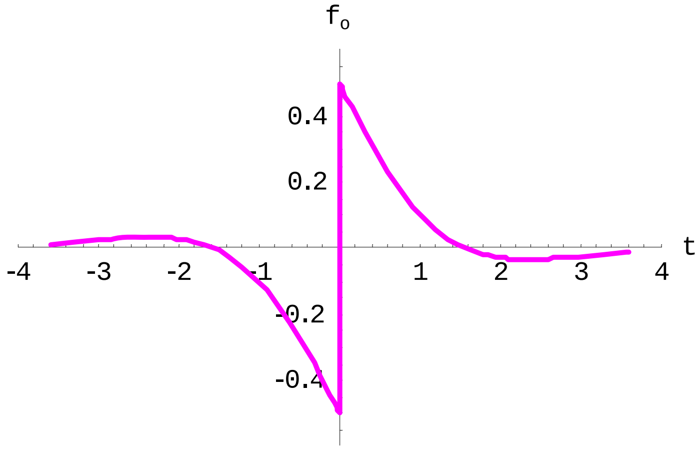
(4) Check its Hilbert transform pair using complex integrals

ANS (1) $f(t)\begin{cases} e^{-t}\cos t & t>0 \\ 0 & t<0 \end{cases}$, $f(-t)\begin{cases} e^{t}\cos t & t<0 \\ 0 & t>0 \end{cases}$

$$f_e(t)=\begin{cases} \frac{1}{2}e^{-t}\cos t & t>0 \\ \frac{1}{2}e^{t}\cos t & t<0 \end{cases}, \quad f_o(t)=\begin{cases} \frac{1}{2}e^{-t}\cos t & t>0 \\ -\frac{1}{2}e^{t}\cos t & t<0 \end{cases}$$

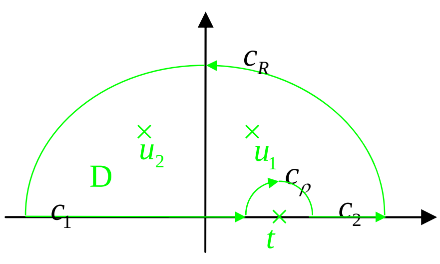
(2)





(3) $F\{f_e(t)+f_o(t)\}=\frac{1}{\sqrt{2\pi}}(\frac{w^2+2-w^3i}{w^4+4})$

(4) $p(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{t-u}du$



$$f(u)=\frac{u^2+2}{u^4+4}=\frac{2+u^2}{(-1+u-i)(-1+u+i)(1+u+i)(1+u-i)}$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{t-u}du=\frac{1}{\pi}\{\int_{C_1+C_2}+\int_{C_\rho}+\int_{C_R} f(u)du\}=2i\sum_{\text{域內}} Res[f(u_1+f(u_2))]$$

$$\text{Limit}[\frac{2+u^2}{(-1+u+i)(1+u+i)(1+u-i)(t-u)},\quad u\to 1+i]+$$

$$\text{Limit}[\frac{2+u^2}{(u-1-i)(u-1+i)(u+1+i)(t-u)},\quad u\to -1+i]$$

$$=\frac{2+t^2-it^3}{2(4+t^4)}$$

$$\text{Limit}[\frac{2+u^2}{(-1+u-i)(-1+u+i)(1+u+i)(1+u-i)},\quad u\to t]$$

$$=\frac{2+t^2}{4+t^4}$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{t-u}du=\frac{1}{\pi}\{\int_{C_1+C_2}+\int_{C_\rho}+\int_{C_R} f(u)du\}=2i\sum_{\text{域內}} Res[f(u_1+f(u_2))]$$

$$\to\frac{1}{\pi}\int_{C_1+C_2}\frac{f(u)}{t-u}du=2i(\frac{2+t^2-it^3}{2(4+t^4)})-i(\frac{2+t^2}{(4+t^4)})$$

$$\to\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(u)}{t-u}du=\frac{t^3}{(4+t^4)}$$

Hint:

$$\int_\rho f(z)dz=\int_\rho\sum_{m=-\infty}^{\infty}a_n(z-w)^n dz=\int_\rho a_{-1}\frac{1}{(z-w)}dz$$

$$\to\int_\pi^0 a_{-1}\frac{1}{\varepsilon e^{it}}i\varepsilon e^{it}dt\quad a_{-1}=\frac{2+t^2}{4+t^4}\qquad\qquad z=w+\varepsilon e^{it}\quad\to\frac{dz}{dt}=i\varepsilon e^{it}$$

$$\to-\pi i(\frac{2+t^2}{4+t^4})$$