Find F, U, V, R, Φ, Ψ and plot deformed shape of a unit square.

(a)

$$\sum = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\Psi^T = \begin{pmatrix} \text{Cos}[\varphi] & -\text{Sin}[\varphi] \\ \text{Sin}[\varphi] & \text{Cos}[\varphi] \end{pmatrix} = \begin{pmatrix} \text{Cos}[-45^\circ] & -\text{Sin}[-45] \\ \text{Sin}[-45] & \text{Cos}[-45^\circ] \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\Phi = \begin{pmatrix} \text{Cos}[\phi] & -\text{Sin}[\phi] \\ \text{Sin}[\phi] & \text{Cos}[\phi] \end{pmatrix} = \begin{pmatrix} \text{Cos}[30^\circ] & -\text{Sin}[30^\circ] \\ \text{Sin}[30^\circ] & \text{Cos}[30^\circ] \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$R = \Phi\,\Psi^T = \begin{pmatrix} \text{Cos}[30^\circ] & -\text{Sin}[30^\circ] \\ \text{Sin}[30^\circ] & \text{Cos}[30^\circ] \end{pmatrix} \begin{pmatrix} \text{Cos}[-45^\circ] & -\text{Sin}[-45] \\ \text{Sin}[-45] & \text{Cos}[-45^\circ] \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{6}+\sqrt{2}}{4} & \frac{\sqrt{6}-\sqrt{2}}{4} \\ \frac{-\sqrt{6}+\sqrt{2}}{4} & \frac{\sqrt{6}+\sqrt{2}}{4} \end{pmatrix}$$

$$F = \Phi \sum \Psi^T = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1+2\sqrt{3}}{4} & \frac{-1+2\sqrt{3}}{4} \\ \frac{2-\sqrt{3}}{4} & \frac{2+\sqrt{3}}{4} \end{pmatrix}$$

$$U = \Psi \sum \Psi^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{3\sqrt{2}}{4} \end{pmatrix}$$

$$V = \Phi \sum \Phi^T = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{7\sqrt{2}}{8} & \frac{\sqrt{6}}{8} \\ \frac{\sqrt{6}}{8} & \frac{5\sqrt{2}}{8} \end{pmatrix}$$

(b)

$$\kappa\,(z) = \frac{1}{2}\,(\sigma_1+\sigma_2)\,\left(e^{i\,(\phi+\varphi)}\,z\right) + \frac{1}{2}\,(\sigma_1-\sigma_2)\,\left(e^{i\,(\phi-\varphi)}\,\bar{z}\right),\quad \phi = \frac{\pi}{6},\quad \varphi = \frac{-\pi}{4},\quad \sigma_1 = \sqrt{2},\quad \sigma_2 = \frac{1}{\sqrt{2}}$$

$$z_1\,(0, 1) = e^{\frac{\pi}{2} i},\quad z_1\,(1, 1) = \sqrt{2}\,e^{\frac{\pi}{4} i},\quad z_1\,(1, 0) = e^{0 i}$$

$$\kappa\,(z) = \frac{1}{2}\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right)\left(e^{i\left(\frac{-\pi}{12}\right)}\,z\right) + \frac{1}{2}\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)\left(e^{i\left(\frac{5\pi}{12}\right)}\,\bar{z}\right)$$

$$\kappa(z_1) = \kappa\left(e^{\frac{\pi}{2}i}\right) = \frac{1}{2}\left(\sqrt{2}+\frac{1}{\sqrt{2}}\right)\left(e^{i\left(\frac{-\pi}{12}\right)} e^{\frac{\pi}{2}i}\right) + \frac{1}{2}\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\left(e^{i\left(\frac{5\pi}{12}\right)} e^{\frac{-\pi}{2}i}\right)$$

$$= \frac{1}{2}\left(\sqrt{2}+\frac{1}{\sqrt{2}}\right)\left(e^{i\left(\frac{5\pi}{12}\right)}\right) + \frac{1}{2}\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\left(e^{i\left(\frac{-\pi}{12}\right)}\right)$$

$$= \frac{2\sqrt{3}-1}{4} + i\,\frac{2+\sqrt{3}}{4}$$

$$\kappa(z_2) = \kappa\left(\sqrt{2}\,e^{\frac{\pi}{4}i}\right) = \frac{1}{2}\left(\sqrt{2}+\frac{1}{\sqrt{2}}\right)\left(e^{i\left(\frac{-\pi}{12}\right)}\left(\sqrt{2}\,e^{\frac{\pi}{4}i}\right)\right) +$$

$$\frac{1}{2}\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\left(e^{i\left(\frac{5\pi}{12}\right)}\left(\sqrt{2}\,e^{\frac{-\pi}{4}i}\right)\right)$$

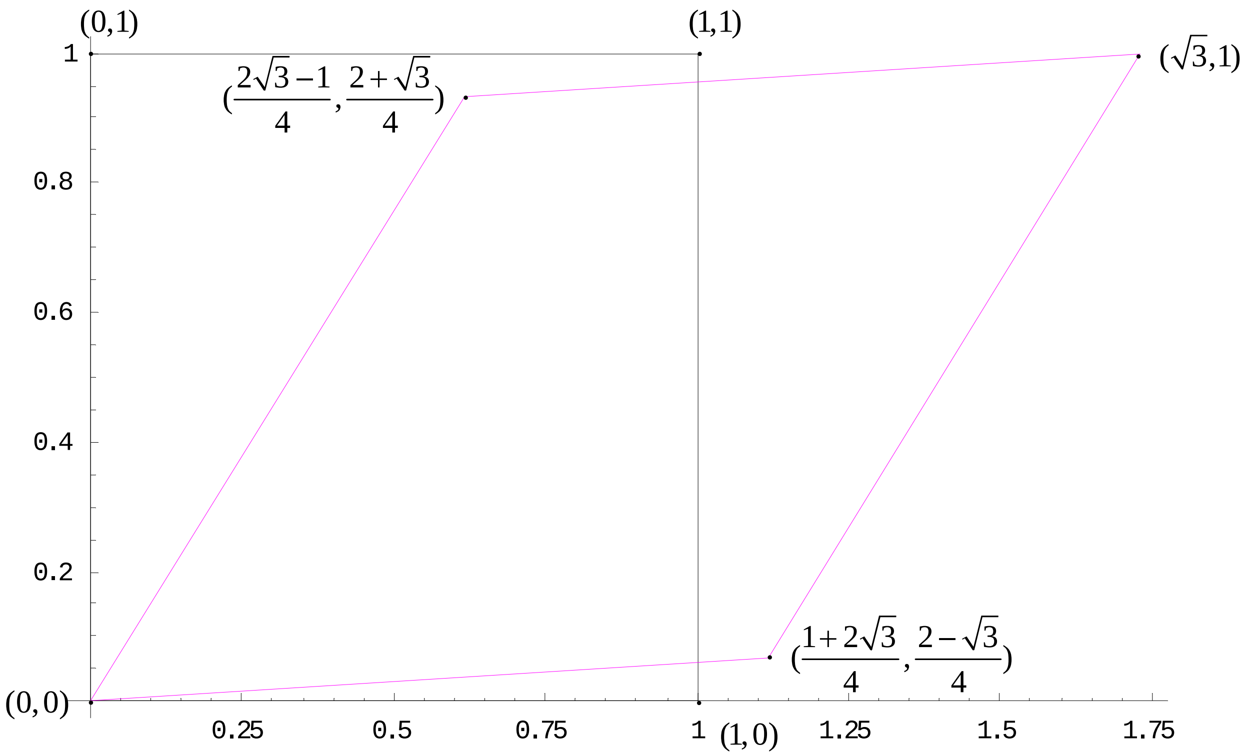
$$= \frac{\sqrt{2}}{2}\left(\sqrt{2}+\frac{1}{\sqrt{2}}\right)\left(e^{i\left(\frac{\pi}{6}\right)}\right) + \frac{\sqrt{2}}{2}\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\left(e^{i\left(\frac{\pi}{6}\right)}\right)$$

$$= \sqrt{3} + i$$

$$\kappa(z_3) = \kappa\left(e^{0i}\right) = \frac{1}{2}\left(\sqrt{2}+\frac{1}{\sqrt{2}}\right)\left(e^{i\left(\frac{-\pi}{12}\right)} e^{0i}\right) + \frac{1}{2}\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\left(e^{i\left(\frac{5\pi}{12}\right)} e^{-0i}\right)$$

$$= \frac{1}{2}\left(\sqrt{2}+\frac{1}{\sqrt{2}}\right)\left(e^{i\left(\frac{-\pi}{12}\right)}\right) + \frac{1}{2}\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\left(e^{i\left(\frac{5\pi}{12}\right)}\right)$$

$$= \frac{1+2\sqrt{3}}{4} + i\,\frac{2-\sqrt{3}}{4}$$



參考解答 by 周克勳