(1), Find the directional derivative of f (x, y) in (1, 0) and (0, 1) directions where f (x, y) = 2 x − 2 y

sol :

$$\vec{f}(x, y) = 2\vec{i} - 2\vec{j}$$

$$\vec{n_1} = \left(1\vec{i} + 0\vec{j}\right), \quad \vec{n_2} = \left(0\vec{i} + 1\vec{j}\right)$$

$$\nabla f \cdot \vec{n_1} = \frac{\partial f}{\partial \vec{n_1}} = \left(2\vec{i} - 2\vec{j}\right) \cdot \left(1\vec{i} + 0\vec{j}\right) = 2$$

$$\nabla f \cdot \vec{n_2} = \frac{\partial f}{\partial \vec{n_2}} = \left(2\vec{i} - 2\vec{j}\right) \cdot \left(0\vec{i} + 1\vec{j}\right) = -2$$

(2), Find the derivative for f (z) = $z^2$ for

(1) $\Delta z = \Delta x$

(2) $\Delta z = i\,\Delta y$

Verify the Cauchy Riemann equation

sol :

$$f(z) = u(x, y) + i\,v(x, y)$$

$$f(z) = z^2 = (x + y\,i)^2 = (x^2 - y^2) + (2\,x\,y)\,i$$

$$u(x, y) = (x^2 - y^2), \quad v(x, y) = (2\,x\,y)$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} =$$

$$\lim_{\Delta x \to 0} \frac{f(z + \Delta x) - f(z)}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) + i\,v(x + \Delta x, y) - u(x, y) - i\,v(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - y^2 + i\,2\,(x + \Delta x)\,y - (x^2 - y^2) - i\,(2\,x\,y)}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{2\,x\,\Delta x + 2\,i\,\Delta x\,y}{\Delta x} = 2\,x + 2\,y\,i = 2\,z$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta y \to 0} \frac{f(z + \Delta y) - f(z)}{i\,\Delta y} =$$

$$\lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) + i\,v(x, y + \Delta y) - u(x, y) - i\,v(x, y)}{i\,\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - (y + \Delta y)^2 + i\,2\,x\,(y + \Delta y) - (x^2 - y^2) - i\,(2\,x\,y)}{i\,\Delta y} =$$

$$\lim_{\Delta x \to 0} \frac{-2\,y\,\Delta y + 2\,i\,x\,\Delta y}{i\,\Delta y} = 2\,x + 2\,y\,i = 2\,z$$

So $f'(z) = 2\,z$

Cauchy Riemann equation :

$$\begin{cases} \frac{\partial u}{\partial x} = 2\,x = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -2\,y = -\frac{\partial v}{\partial x} \end{cases}$$

(3), Find the derivative for f (z) = $\bar{z}$ for

(1) $\Delta z = \Delta x$

(2) $\Delta z = i\,\Delta y$

Verify the Cauchy Riemann equation

sol :

$$f(z) = u(x, y) + i\,v(x, y)$$

$$f(z) = \bar{z} = x - y\,i\,\bar{-}^2 = x - y\,i$$

$$u(x, y) = x, \quad v(x, y) = -y$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} =$$

$$\lim_{\Delta x \to 0} \frac{f(z + \Delta x) - f(z)}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) + i\,v(x + \Delta x, y) - u(x, y) - i\,v(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - i\,y - x + i\,y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} =$$

$$\lim_{\Delta y \to 0} \frac{f(z+\Delta y) - f(z)}{i\,\Delta y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + i\,v(x, y+\Delta y) - u(x, y) - i\,v(x, y)}{i\,\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{x - i(y+\Delta y) - x + i\,y}{i\,\Delta y} = \lim_{\Delta x \to 0} \frac{-i\,\Delta y}{i\,\Delta y} = -1$$

**So $f'(z)$ = 不存在**

**Cauchy Riemann equation :**

$$\begin{cases} \frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1 \\ \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x} \end{cases}$$

**(4), Find the derivative for $f(z) = \text{Re}\{z\}$ for**

(1) $\Delta z = \Delta x$

(2) $\Delta z = i\,\Delta y$

**Verify the Cauchy Riemann equation**

**sol :**

$$f(z) = u(x, y) + i\,v(x, y)$$

$f(z) = \text{Re}\{z\} = x$

$u(x, y) = x,\ v(x, y) = 0$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta y \to 0} \frac{f(z+\Delta y) - f(z)}{i\,\Delta y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x, y)}{i\,\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{x - x}{i\,\Delta y} = \lim_{\Delta x \to 0} \frac{0}{i\,\Delta y} = 0$$

**So $f'(z)$ = 不存在**

**Cauchy Riemann equation :**

$$\begin{cases} \frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 0 \\ \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x} \end{cases}$$

**(5), Find the derivative for $f(z) = \text{Im}\{z\}$ for**

(1) $\Delta z = \Delta x$

(2) $\Delta z = i\,\Delta y$

**Verify the Cauchy Riemann equation**

**sol :**

$$f(z) = u(x, y) + i\,v(x, y)$$

$f(z) = \text{Im}\{z\} = y$

$u(x, y) = y,\ v(x, y) = 0$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{y - y}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta y \to 0} \frac{f(z+\Delta y) - f(z)}{i\,\Delta y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x, y)}{i\,\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{(y+\Delta y) - y}{i\,\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta y}{i\,\Delta y} = -i$$

**So $f'(z)$ = 不存在**

**Cauchy Riemann equation :**

$$\begin{cases} \frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = 1 \neq -\frac{\partial v}{\partial x} = 0 \end{cases}$$