Engineering Mathematics(III)

Department of Harbor and River Engineering, National Taiwan Ocean University
due 30/12 morning, Fall 1994, J. T. Chen

HW No.1 Different points of view for Poisson summation formula:
***** Integral equations ****** BEM Text book by J. T. Chen (page 123) *******
[範例]：試以映像法，推導Poisson 積分公式。

控制方程 $\nabla^2 u(r,\theta) = 0, \;\; 0 < r < R, \;\; 0 < \theta < 2\pi$

邊界條件 $u(r,\theta) = g(\theta), \;\; for \;\; r = R$

滿足上式的解可以寫成

$$2\pi\phi(s) = \int_B U(x;s,s')t(x)dB(x) - \int_B T(x;s,s')u(x)dB(x)$$

若取

$$U(x;s,s') = U(x,s) + U(x,s') + U(s')$$

其中

$$U(x,s) = -ln \mid x - s \mid$$

$$U(x,s') = ln \mid x - s' \mid$$

$$U(s') = -ln(\frac{R}{\sqrt{s_1^2 + s_2^2}})$$

Poisson 核對應的映像奇異系統

由上圖幾何關係可知

$$x_1 = \bar{\rho}\; cos(\bar{\theta}), \quad x_2 = \bar{\rho}\; sin(\bar{\theta})$$

$$s_1 = \rho\; cos(\theta), \quad s_2 = \rho\; sin(\theta)$$

$$\theta_1 = \bar{\theta} - \theta$$

$$r^2 = \rho^2 + \bar{\rho}^2 - 2\rho\bar{\rho}cos(\theta_1)$$

$$r'^2 = \bar{\rho}^2 + (\frac{R^2}{\rho})^2 - 2R^2\frac{\bar{\rho}}{\rho}cos(\theta_1)$$

因此

$$U(x;s,s') = -ln(r) + ln(r') - ln(\rho/R)$$

當$\bar{\rho} = R$ 時，意即x 在$R = 1$ 的圓上，所以

$$U(x;s,s') = 0$$

解的積分表示式，更可精簡的寫成

$$2\pi\phi(s) = -\int_B T(x;s,s')u(x)dB(x)$$

而若x 在圓周上，則

$$T(x;s,s') = \frac{-1}{2\pi R}\frac{(R^2-\rho^2)}{[R^2+\rho^2-2R\rho\ cos(\theta_1)]}$$

解為

$$u(s) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-\rho^2)}{[R^2+\rho^2-2R\rho\ cos(\theta)]}g(\theta)d\theta$$

上式即為很有名的Poisson 積分公式。而

$$k(s,x) = \frac{1}{2\pi}\frac{(R^2-\rho^2)}{[R^2+\rho^2-2R\rho\ cos(\theta)]}$$

其中，$k(s,x)$ 即為Poisson 核函數。

***** Complex integration ***********************************

(a). By setting $z = Re^{i\theta}$ and $z_0 = \rho e^{i\phi}$, show that the Cauchy integral formula:

$$f(\rho e^{i\phi}) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(Re^{i\theta})}{(Re^{i\theta}-\rho e^{i\phi})}iRe^{i\theta}d\theta$$

(b). Show that

$$0 = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(Re^{i\theta})}{(Re^{i\theta}-R^2e^{i\phi}/\rho)}iRe^{i\theta}d\theta, \rho < R$$

(c). Derive the Poisson summation formula.

$$u(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-\rho^2)}{[R^2+\rho^2-2R\rho\ cos(\theta-\phi)]}g(\theta)d\theta, f(z) = u+iv$$

***** Moment generating function *****************************

$$\frac{1-r^2}{2(r^2+1-2rcos(\theta))} = \frac{1}{2}+\sum_1^\infty r^n cos(n\theta)$$