工程數學（三）—複變　期末作業

海大河海系　　　陳正宗

HW No.1 Different points of view for Poisson summation formula:

***** Integral equations ****** BEM Text book by J. T. Chen (page 123) *******

[範例]：對於內域問題，上次作業已推導得Poisson 積分公式

一、映像法***** Image method ************************************

控制方程　$\nabla^2 u(r,\theta) = 0,\ \ 0 < r < R,\ \ 0 < \theta < 2\pi$

邊界條件　$u(r,\theta) = g(\theta),\ for\ r = R$

解為

$$u(s) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-\rho^2)}{[R^2+\rho^2-2R\rho\ cos(\theta)]} g(\theta) d\theta$$

上式即為很有名的Poisson 積分公式。而

$$k(s,x) = \frac{1}{2\pi}\frac{(R^2-\rho^2)}{[R^2+\rho^2-2R\rho\ cos(\theta)]}$$

其中，$k(s,x)$ 即為Poisson 核函數。

二、複數積分法***** Complex integration ************************************

(a). By setting $z = Re^{i\theta}$ and $z_0 = \rho e^{i\phi}$, show that the Cauchy integral formula:

$$f(\rho e^{i\phi}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(Re^{i\theta})}{(Re^{i\theta}-\rho e^{i\phi})} iRe^{i\theta} d\theta$$

(b). Show that

$$0 = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(Re^{i\theta})}{(Re^{i\theta}-R^2 e^{i\phi}/\rho)} iRe^{i\theta} d\theta, \rho < R$$

(c). Derive the Poisson summation formula.

$$u(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-\rho^2)}{[R^2+\rho^2-2R\rho\ cos(\theta-\phi)]} g(\theta) d\theta, f(z) = u + iv$$

三、 Fourier 係數法 ***** Moment generating function *****************************

$$\frac{1-r^2}{2(r^2+1-2rcos(\theta))} = \frac{1}{2} + \sum_1^{\infty} r^n cos(n\theta)$$

四、保角寫像***** Conformal mapping ******************************

習題一: 將上式的內域Poisson 積分公式改成外域Poisson 積分公式，則將有何改變?

$$u(\rho,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-R^2}{R^2-\rho^2-2R\rho\ cos(\theta-\theta')}g(\theta')d\theta'$$

試以以下四種方法，分別推導

一、映像法 (積分型式解)、

二、複數積分法 (積分型式解)、

三、 Fourier 係數法 (級數型式解)、

四、保角寫像 (確切解)

1. Consider the following problem:

   Governing equation:

   $$\nabla^2 u(r,\theta)=0,\ \ R<r<\infty,\ \ 0<\theta<2\pi$$

   Boundary condition:

   $$u(r,\theta)=g(\theta),\ for\ r=R$$

   Please derive the Poisson formula for exterior domain.

   $$u(\rho,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2-R^2}{R^2-\rho^2-2R\rho\ cos(\theta-\theta')}g(\theta')d\theta'$$

2. Solve the above exterior problem either analytically or numerically for the following B.C.

   $$g(\theta)=\pm1.0,\ +\ for\ \ 0<\theta<\pi,\ -\ for\ \ \pi<\theta<2\pi$$

   where the radius is $R=1$.

3. Plot the potential and potential gradient along the three angles 30, 60, 90 degrees from $\rho=1$ to $\rho=5$. Also, plot the normal flux on the circular boundary.

4. Based on the three point transformation, derive the exact solution:

   $$u(x,y)=\frac{2}{\pi}tan^{-1}(\frac{2y}{x^2+y^2-1})$$