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AN ALTERNATIVE DERIVATION OF THE POLAR DECOMPOSITION

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Introduction

The deformation process at a point due to external traction may be considered as the result of a translation followed by the rotation of the principal axes of strains and stretches along the principal axes or vice versa. This idea, known as polar decomposition theorem, is that the deformation gradient tensor **F** is decomposed into two tensors, **F** = **RU** (or **F** = **VR**). The tensor **R** represents a proper orthogonal rotation tensor (skew-symmetric). **U** and **V** are, respectively, defined as the right stretch tensor and the left stretch tensor which are symmetric positive definite. Malvern[1] derived polar decomposition from the difference of the final squared length $(ds)^2$ of the element in the deformed configuration and the initial squared length $(dS)^2$ of the element in the undeformed configuration.

This study presents the derivation procedure of polar decomposition in a different way on the basis of Continuum Mechanics. The polar decomposition has been derived systematically from the deformation gradient tensor **F** itself. The results could be potentially valuable from the point of view of the mathematical manipulations of the deformation gradient tensor **F**.

The polar decomposition theorem

The general motion of a continuum is described by $\boldsymbol{x} = \boldsymbol{x}(\mathbf{X},t)$ where $\boldsymbol{x}$ is the spatial position vector at time t, of a material particle with a material coordinate **X**. Deformation at a material point **X** of a body is characterized by changes of distances between any pair of material points within the small neighborhood of **X**. A material element $d\mathbf{X}$ at the reference configuration is transformed, through motion, into a material element dx at time t. The relation between $d\mathbf{X}$ and dx is given by $dx = \mathbf{F}\,d\mathbf{X}$, where the tensor **F** is called the deformation gradient at **X**. It has been known that for any real tensor **F** with a non-zero determinant (i.e., $\mathbf{F}^{-1}$ exists), one can always decompose it into the product of a proper orthogonal tensor and a positive definite symmetric tensor; **F** = **RU** or (**F** = **VR**).

Derivation of polar decomposition in a different way

This study demonstrates that the polar decomposition theorem can be derived in a different method, compared with Malvern's method[1]. On the following two steps, the polar decomposition is derived systematically from the deformation gradient tensor **F** itself.

Step 1: Since $\mathbf{F}^T\mathbf{F}$ is a positive definite symmetric tensor, the square root tensor, $\sqrt{\mathbf{F}^T\mathbf{F}} = \mathbf{U}$, defined as the right stretch tensor always exists [1]. In addition, the square root tensor itself is also positive definite symmetric tensor. Hence, the deformation gradient tensor **F** always can be expressed as

$$\mathbf{F} = \mathbf{F}\mathbf{U}^{-1}\mathbf{U}. \tag{1}$$

where $(\ )^{-1}$ represents inverse of tensor. Equation (1) could be rewritten as

$$\mathbf{F} = \left(\mathbf{F}\mathbf{U}^{-1}\right)\mathbf{U}, \tag{2}$$

Equation (2) represents that deformation gradient tensor **F** is decomposed into the product of two tensors; $(\mathbf{F}\mathbf{U}^{-1})$ and **U**. **U** is the right stretch tensor and $(\mathbf{F}\mathbf{U}^{-1})$ is any real tensor.
Taking determinant on $\mathbf{F}^T\mathbf{F}$ yields that

$$\det[\mathbf{F}^T\mathbf{F}] = \det[\mathbf{U}^2]$$

$$\det[\mathbf{F}^T]\det[\mathbf{F}] = \det[\mathbf{U}]\det[\mathbf{U}], \tag{3}$$

where det [ ] represents determinant of tensor. Subsequently,

$$\det[\mathbf{U}] = \det[\mathbf{F}] = \det[\mathbf{F}^T]. \tag{4}$$

By similar manner,

$$\det[\mathbf{V}] = \det[\mathbf{F}^T] = \det[\mathbf{F}]. \tag{5}$$

Step 2: Next let us look at the tensor ($\mathbf{FU}^{-1}$) in (2) carefully. If we take transpose on $\mathbf{FU}^{-1}$ and post multiply by $\mathbf{FU}^{-1}$, then

$$\begin{aligned}(\mathbf{FU}^{-1})^T(\mathbf{FU}^{-1}) &= (\mathbf{U}^{-1})^T\mathbf{F}^T\mathbf{FU}^{-1}\\ &= (\mathbf{U}^{-1})^T\mathbf{U}^2\mathbf{U}^{-1}\\ &= (\mathbf{U}^{-1})^T\mathbf{U}\,\mathbf{U}\,\mathbf{U}^{-1}\\ &= (\mathbf{U}^{-1})^T\mathbf{U}^T\mathbf{U}\,\mathbf{U}^{-1}\\ &= (\mathbf{U}\,\mathbf{U}^{-1})^T\mathbf{U}\,\mathbf{U}^{-1}\\ &= \mathbf{I}. \end{aligned} \tag{6}$$

Taking determinant on $\mathbf{FU}^{-1}$ yields that

$$\begin{aligned}\det[\mathbf{FU}^{-1}] &= \det[\mathbf{F}]\det[\mathbf{U}^{-1}]\\ &= \det[\mathbf{F}]\frac{1}{\det[\mathbf{U}]}\\ &= \det[\mathbf{U}]\frac{1}{\det[\mathbf{U}]}\\ &= +1. \end{aligned} \tag{7}$$

Here, (6) and (7) show that $\mathbf{FU}^{-1}$ is nothing but the proper orthogonal rotation tensor $\mathbf{R}$. Recall that $\mathbf{R}$ is the proper orthogonal rotation tensor if $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ and $\det[\mathbf{R}] = +1$. Therefore $\mathbf{F} = (\mathbf{FU}^{-1})\mathbf{U} = \mathbf{RU}$.

The derivation for $\mathbf{F} = \mathbf{VR}$ also can be obtained by similar manner. Since $\mathbf{FF}^T$ is positive definite symmetric tensor, the square root tensor, $\sqrt{\mathbf{FF}^T} = \mathbf{V}$, defined as the left stretch tensor always exists [1]. Then, the deformation gradient tensor $\mathbf{F}$ always can be expressed as

$$\mathbf{F} = \mathbf{VV}^{-1}\mathbf{F}. \tag{8}$$

Equation (8) could be rewritten as

$$\mathbf{F} = \mathbf{V}\left(\mathbf{V}^{-1}\mathbf{F}\right). \tag{9}$$

Equation (9) represents that tensor $\mathbf{F}$ is decomposed into the product of two tensor; $\mathbf{V}$ and ($\mathbf{V}^{-1}\mathbf{F}$). $\mathbf{V}$ is the left stretch tensor and ($\mathbf{V}^{-1}\mathbf{F}$) is any real tensor. If we take transpose on $\mathbf{V}^{-1}\mathbf{F}$ and post multiply by $\mathbf{V}^{-1}\mathbf{F}$, then

$$\begin{aligned}(\mathbf{V}^{-1}\mathbf{F})^T(\mathbf{V}^{-1}\mathbf{F}) &= \mathbf{F}^T(\mathbf{V}^{-1})^T(\mathbf{V}^{-1})\mathbf{F}\\ &= \mathbf{F}^T(\mathbf{V}^T)^{-1}\mathbf{V}^{-1}\mathbf{F}\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{F}^T \mathbf{V}^{-1} \mathbf{V}^{-1} \mathbf{F} \\
&= \mathbf{F}^T (\mathbf{V}\mathbf{V})^{-1} \mathbf{F} \\
&= \mathbf{F}^T (\mathbf{V}^2)^{-1} \mathbf{F} \qquad (10) \\
&= \mathbf{F}^T (\mathbf{F}\mathbf{F}^T)^{-1} \mathbf{F} \\
&= \mathbf{F}^T (\mathbf{F}^T)^{-1} \mathbf{F}^{-1} \mathbf{F} \\
&= \mathbf{I}.
\end{aligned}
$$

Taking determinant on $\mathbf{V}^{-1}\mathbf{F}$ gives that

$$
\begin{aligned}
\det[\mathbf{V}^{-1}\mathbf{F}] &= \det[\mathbf{V}^{-1}] \det[\mathbf{F}] \\
&= \frac{1}{\det[\mathbf{V}]} \det[\mathbf{F}] \\
&= \frac{1}{\det[\mathbf{V}]} \det[\mathbf{V}] \\
&= +1. \qquad (11)
\end{aligned}
$$

Here again, (10) and (11) show that $\mathbf{V}^{-1}\mathbf{F}$ is nothing but the proper orthogonal rotation tensor $\mathbf{R}$. Therefore $\mathbf{F} = \mathbf{V}(\mathbf{V}^{-1}\mathbf{F}) = \mathbf{V}\mathbf{R}$.

Summary

This paper has showed that there is an alternative in deriving polar decomposition, which is quite different from Malvern's method[1]. Derivation procedure presented in this study has been started from the deformation gradient tensor **F** itself. Meanwhile, derivation procedure by Malvern [1] was started from the difference of the final squared length $(ds)^2$ of the element in the deformed configuration and the initial squared length $(dS)^2$ of the element in the undeformed configuration.

References

1. L.E. Malvern, Introduction to the Mechanics of a Continuous Medium, p. 178. Prentice - Hall, Inc, NJ (1969).
2. W.M. Lai, D. Rubin and E. Krempl, Introduction to Continuum Mechanics, 3rd ed., p. 124. Pergamon Press Inc., NY (1993).