一階微分方程解法

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linear first order ODE

$$\dot{y}(x) + a(x)y(x) = f(x)$$

Linearity :

$$\dot{y}_1(x) + a(x)y_1(x) = 0$$
$$\dot{y}_2(x) + a(x)y_2(x) = 0$$
$$\dot{Y}(x) + a(x)Y(x) = 0$$
$$Y(x) = y_1(x) + y_2(x)$$

y_1 and y_2 are solutions $\rightarrow Y(x)$ is solution.

Method 1: integration factor for $a(x) = a$ only

$$e^{ax}\dot{y}(x) + e^{ax}ay(x) = e^{ax}f(x)$$
$$\frac{d\{e^{ax}y(x)\}}{dx} = e^{ax}f(x)$$
$$e^{ax}y(x) = \int e^{ax}f(x)dx + c$$
$$y(x) = e^{-ax}\int e^{ax}f(x)dx + ce^{-ax}$$

Method 1: integration factor for $a(x)$ only

$$e^{\int a(x)dx}\dot{y}(x) + e^{\int a(x)dx}a(x)y(x) = e^{\int a(x)dx}f(x)$$
$$\frac{d\{e^{\int a(x)dx}y(x)\}}{dx} = e^{\int a(x)dx}f(x)$$
$$e^{\int a(x)dx}y(x) = \int e^{\int a(x)dx}f(x)dx + c$$
$$y(x) = e^{-\int a(x)dx}\int e^{\int a(x)dx}f(x)dx + ce^{-\int a(x)dx}$$

Method 2: variation of parameters $a(x)$ only

$$y(x) = u(x)v(x)$$
$$uv' + u'v + a(x)uv = f(x)$$
$$uv' + (u' + a(x)u)v = f(x)$$
$$uv' = f(x)$$
$$(u' + a(x)u) = 0$$
$$u(x) = c_1e^{-\int a(x)dx}$$
$$v' = u^{-1}f(x) = \frac{1}{c_1}e^{\int a(x)dx}f(x)$$
$$v(x) = \frac{1}{c_1}\int e^{\int a(x)dx}f(x)dx + c_2$$
$$y(x) = e^{-\int a(x)dx}\{\int e^{\int a(x)dx}f(x)dx\} + ce^{-\int a(x)dx}$$