變數分離在微分方程的應用

海大河工系 陳正宗

Separation of space and time

x:space, t:time, $X_i(x)$:mode, $T_i(t)$: generalized coordinate

$$f(x,t) = X(x)T(t)$$

$$f(x,t) = \sum_{i=0}^{N} X_i(x)T_i(t)$$

$$f(x,t) = \sum_{i=0}^{\infty} X_i(x)T_i(t)$$

Example:

$$f(x,t) = xt$$

$$f(x,t) = sin(x-t) = sin(x)cos(t) - cos(x)sin(t)$$

$$f(x,t) = \frac{1}{\sqrt{1+t^2-2tx}} = \sum_{i=0}^{\infty} t^i P_i(x)$$

$$f(x,t) = \frac{1}{\sqrt{t-x}} =?$$

Example:

$$\frac{dy(x)}{dx} = f(x,y) = X(x)Y(y)$$

$$\int \frac{dy}{Y(y)} = \int X(x)dx + c$$

case 1:

$$\frac{dx}{dt} = t\sqrt{1-x^2}$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \int tdt + c \quad \rightarrow \quad x(t) = sin(\frac{t^2}{2} + c)$$

case 2: Escape velocity

$$\ddot{r}(t) = -g\frac{R^2}{r^2}$$

$$\dot{v}(t) = \frac{dv(t)}{dt} = -g\frac{R^2}{r^2}$$

$$\dot{v}(t) = \frac{dv(t)}{dr}\frac{dr}{dt} = -g\frac{R^2}{r^2}$$

$$\dot{v}(t) = \frac{dv(t)}{dr}v = -g\frac{R^2}{r^2}$$

$$v^2 = \frac{2gR^2}{r} + C$$

At time $t = 0$,

$$t = 0, r(0) = R, v(0) = v_0 \quad \rightarrow \quad C = -2gR + v_0^2$$

Escape velocity, $r = \infty, v = 0$ to determine the v_0

$$v_0 = \sqrt{2gR}$$