國立臺灣海洋大學河海工程學系 2001 工程數學 (三) 第四次作業小考參考解答

1. $f(x) = \begin{cases} 0 & -1 < x < 0 \\ x & 0 < x < 1 \end{cases}$, $x \in [-1, 1]$

(a) please use the Fourier series to expand $f(x)$ and plot the function.

ie.

$$y(x) = \sum_{n=0}^{6}[a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T})]$$

where T=2

Sol:

$$y(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(\frac{2n\pi x}{T}) + b_n \sin(\frac{2n\pi x}{T})]$$

where $T = 2$

$$a_0 = \frac{1}{2}\int_{-1}^{1} 1\ xdx = \frac{1}{2}\int_{0}^{1} xdx = \frac{1}{4}$$

$$\begin{aligned} a_n &= \frac{2}{2}\int_{-1}^{1} x\cos(n\pi x)dx \\ &= \int_{0}^{1} x\cos(n\pi x)dx \\ &= [\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{(n\pi)^2}\cos(n\pi x)]|_0^1 \\ &= \frac{1}{(n\pi)^2}\cos(n\pi) - \frac{1}{(n\pi)^2} \\ &= \frac{1}{(n\pi)^2}[(-1)^n - 1] \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{2}\int_{-1}^{1} x\sin(n\pi x)dx \\ &= \int_{0}^{1} x\sin(n\pi x)dx \\ &= [-\frac{x}{n\pi}\cos(n\pi x) + \frac{1}{(n\pi)^2}\sin(n\pi x)]|_0^1 \\ &= -\frac{1}{n\pi}\cos(n\pi) = -\frac{(-1)^n}{n\pi} \end{aligned}$$

$$y_n(x) = \frac{1}{4} + \sum_{n=1}^{3}[\frac{(-1)^n - 1}{(n\pi)^2}\cos(n\pi x) + \frac{-(-1)^n}{n\pi}\sin(n\pi x)]$$

$$\underline{y_0(x) = \frac{1}{4}}$$

$$\underline{y_1(x) = \frac{1}{4} - \frac{2}{\pi^2}\cos(\pi x)}$$

$$\underline{y_2(x) = \frac{1}{4} - \frac{2}{\pi^2}\cos(\pi x) + \frac{1}{\pi}\sin(\pi x)}$$

$$\underline{y_3(x) = \frac{1}{4} - \frac{2}{\pi^2}\cos(\pi x) + \frac{1}{\pi}\sin(\pi x)}$$

$$y_4(x) = \frac{1}{4} - \frac{2}{\pi^2}\cos(\pi x) + \frac{1}{\pi}\sin(\pi x) - \frac{1}{2\pi}\sin(2\pi x)$$

(b) Please use the Legendre polynomial to expand $f(x)$ and plot the function.

ie.

$$y(x) = \sum_{i=0}^{6} C_i P_i(x)$$

where $P_i(x)$ is an i th order Legendre polynomial.

Sol:

$$y(x) = \sum_{i=0}^{\infty} C_i P_i(x) = \sum_{i=0}^{\infty}\left(\frac{\int_{-1}^{1} P_i(x) f(x) dx}{\int_{-1}^{1} P_i^2(x) dx}\right)\left(\frac{1}{2^i i!}\frac{d^i}{dx^i}(x^2-1)^i\right)$$

$$C_0 = \frac{\int_{-1}^{1}(1)x dx}{\int_{-1}^{1}(1)^2 dx} = \frac{1}{4}$$

$$C_1 = \frac{\int_{-1}^{1}(x)x dx}{\int_{-1}^{1}(x)^2 dx} = \frac{1}{2}$$

$$C_2 = \frac{\int_{-1}^{1}[\frac{1}{2}(3x^2-1)]x dx}{\int_{-1}^{1}[\frac{1}{2}(3x^2-1)]^2 dx} = \frac{5}{16}$$

$$C_3 = \frac{\int_{-1}^{1}[\frac{1}{2}(5x^3-3x)]x dx}{\int_{-1}^{1}[\frac{1}{2}(5x^3-3x)]^2 dx} = 0$$

$$C_4 = \frac{\int_{-1}^{1}[\frac{1}{8}(35x^4 - 30x^2 + 3)x]dx}{\int_{-1}^{1}[\frac{1}{2}(35x^4 - 30x^2 + 3)]^2 dx} = -\frac{3}{32}$$

$$C_5 = \frac{\int_{-1}^{1}[\frac{1}{8}(63x^5 - 70x^3 + 15x)]xdx}{\int_{-1}^{1}[\frac{1}{2}(63x^5 - 70x^3 + 15x)]^2 dx} = 0$$

$$C_6 = \frac{\int_{-1}^{1}[\frac{1}{8}(35x^4 - 30x^2 + 3)]xdx}{\int_{-1}^{1}[\frac{1}{2}(35x^4 - 30x^2 + 3)]^2 dx} = -\frac{3}{32}$$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

$$y_i(x) = \sum_{i=0}^{6} C_i P_i(x)$$

$$\underline{y_0(x) = \frac{1}{4}}$$

$$\underline{y_1(x) = \frac{1}{4} + \frac{1}{2}x}$$

$$\underline{y_2(x) = \frac{3}{32} + \frac{1}{2}x + \frac{15}{32}x^2}$$

$$\underline{y_3(x) = \frac{3}{32} + \frac{1}{2}x + \frac{15}{32}x^2}$$

$$\underline{y_4(x) = \frac{15}{256} + \frac{1}{2}x + \frac{105}{120}x^2 - \frac{105}{256}x^4}$$

$$\underline{y_5(x) = \frac{15}{256} + \frac{1}{2}x + \frac{105}{120}x^2 - \frac{105}{256}x^4}$$

$$\underline{y_6(x) = \frac{25}{256} + \frac{1}{2}x + \frac{525}{128}x^2 - \frac{2625}{256}x^4 + \frac{231}{32}x^6}$$

2. Compare the accuracy of Fourier series and Legendre polynomial.

Sol:

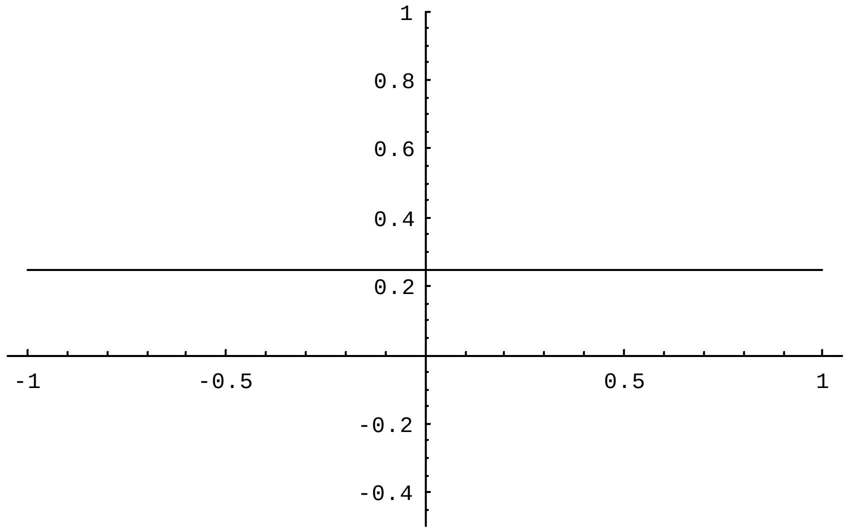
(a)Fourier series approach

$$\int_{-1}^{1}[f(x)-y_6(x)]^2dx = \int_{-1}^{0}[f(x)-y_6(x)]^2dx + \int_{-1}^{1}[f(x)-y_6(x)]^2dx = 0.0288531$$

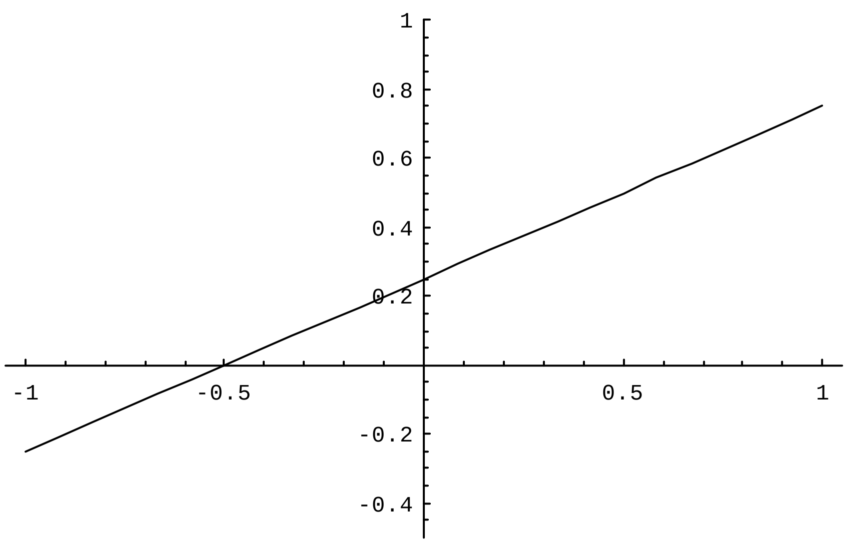
(b)Legendre polynomial approach

$$\int_{-1}^{1}[f(x)-y_6(x)]^2dx = \int_{-1}^{0}[f(x)-y_6(x)]^2dx + \int_{-1}^{1}[f(x)-y_6(x)]^2dx = 0.000254313$$

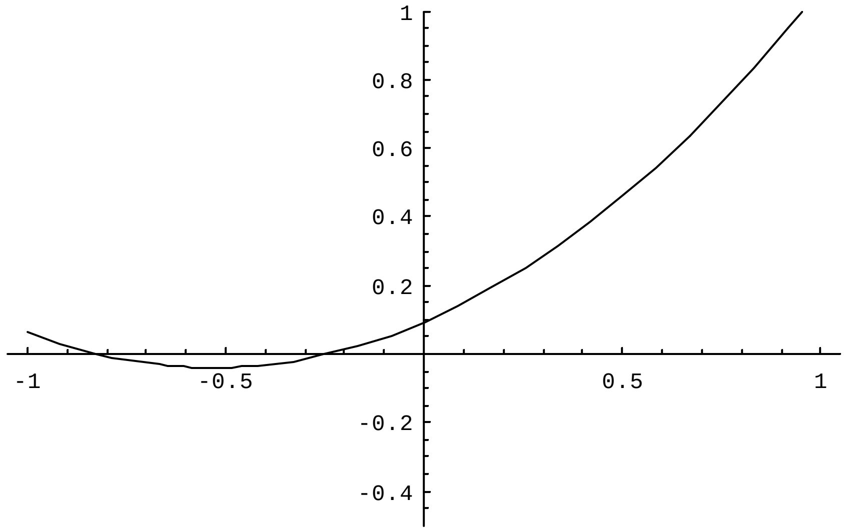
Compare with the results of (a) and (b) getting the accuracy of Legendre polynomial is better then that of Fourier series.



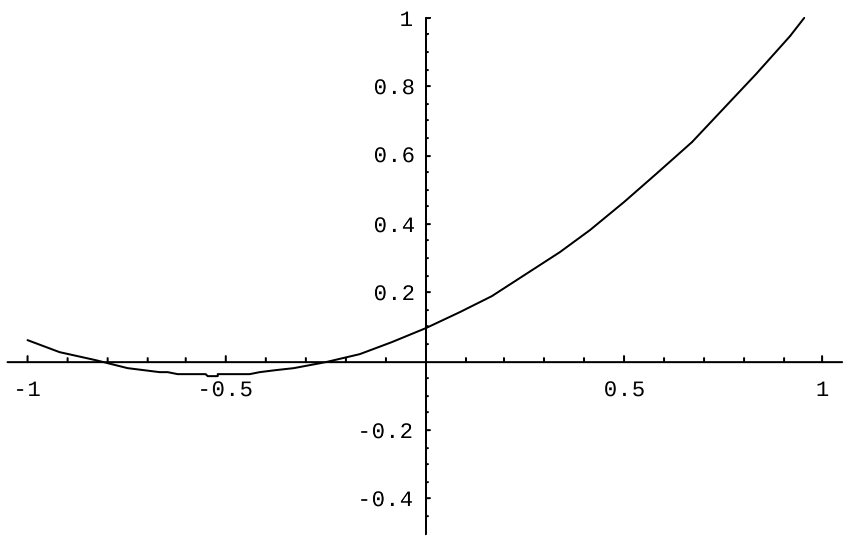
FIGGURE 1. : Legendre polynomial $y_0 = \sum_{i=0}^{0} c_i P_i$



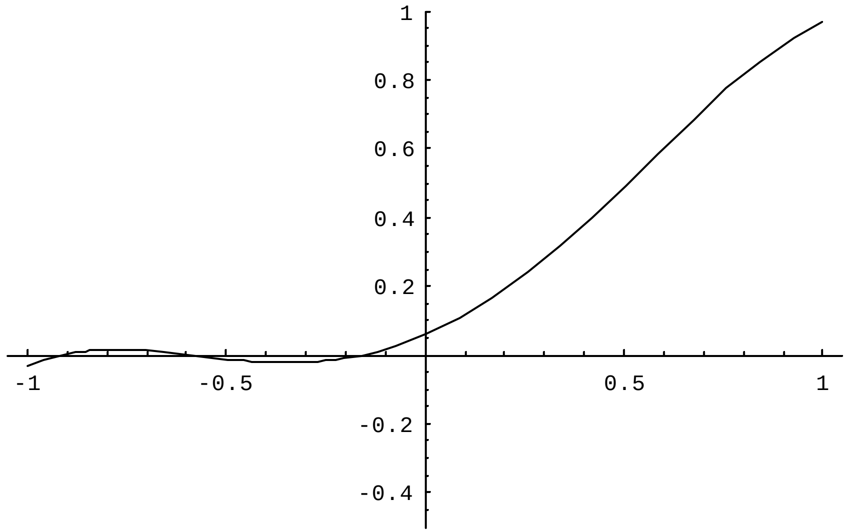
FIGGURE 2. : Legendre polynomial $y_1 = \sum_{i=0}^{1} c_i P_i$



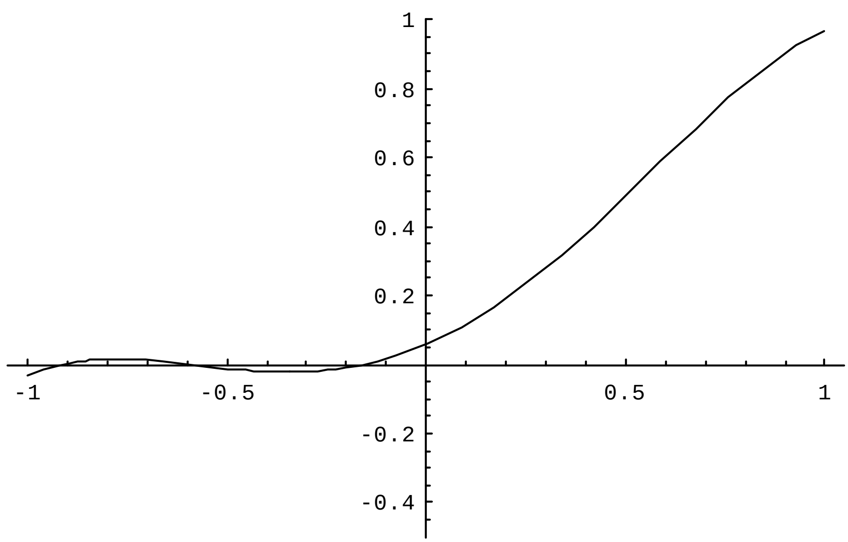
FIGGURE 3. : Legendre polynomial $y_2 = \overset{2}{\underset{i=0}{\hat{a}}} c_i P_i$



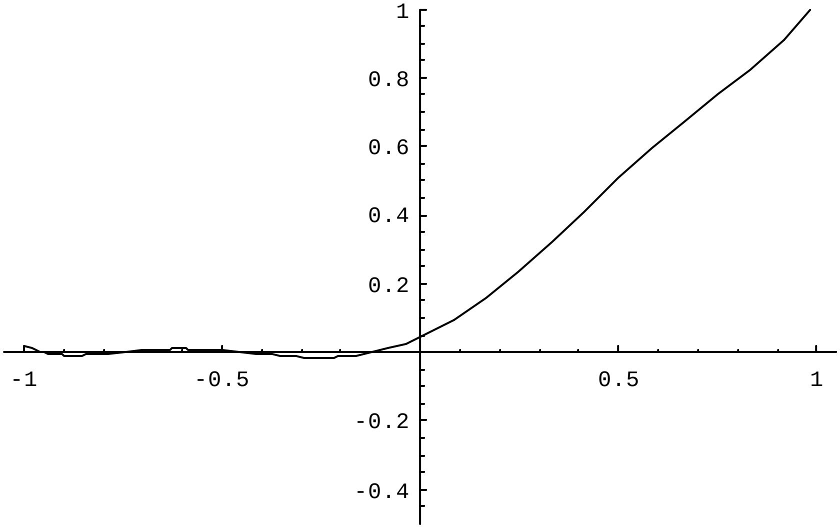
FIGGURE 4. : Legendre polynomial $y_3 = \overset{3}{\underset{i=0}{\hat{a}}} c_i P_i$



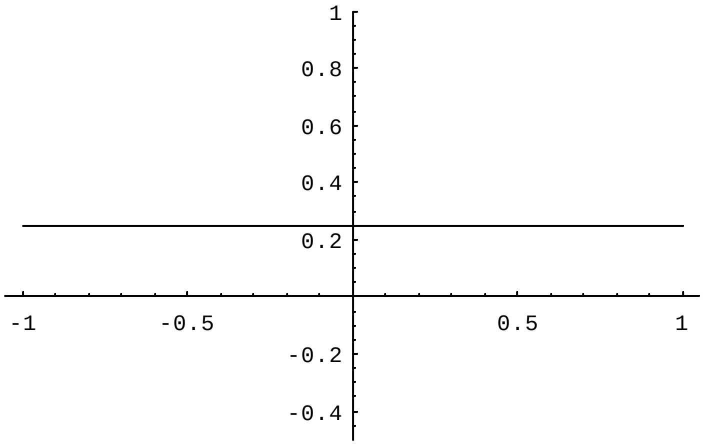
FIGGURE 5. : Legendre polynomial $y_4 = \sum_{i=0}^{4} c_i P_i$



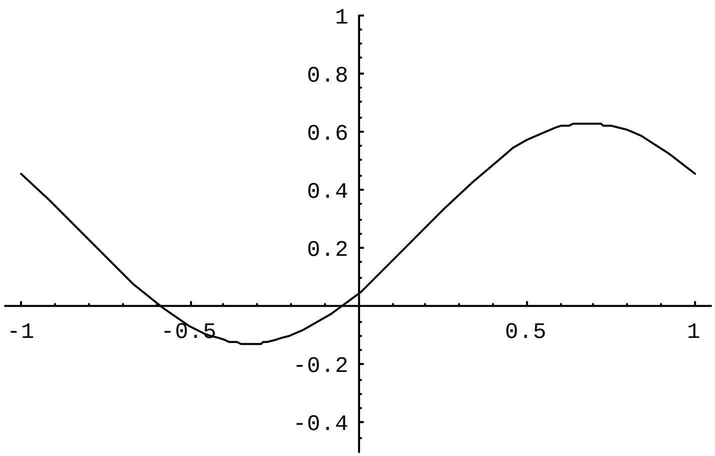
FIGGURE 6. : Legendre polynomial $y_5 = \sum_{i=0}^{5} c_i P_i$



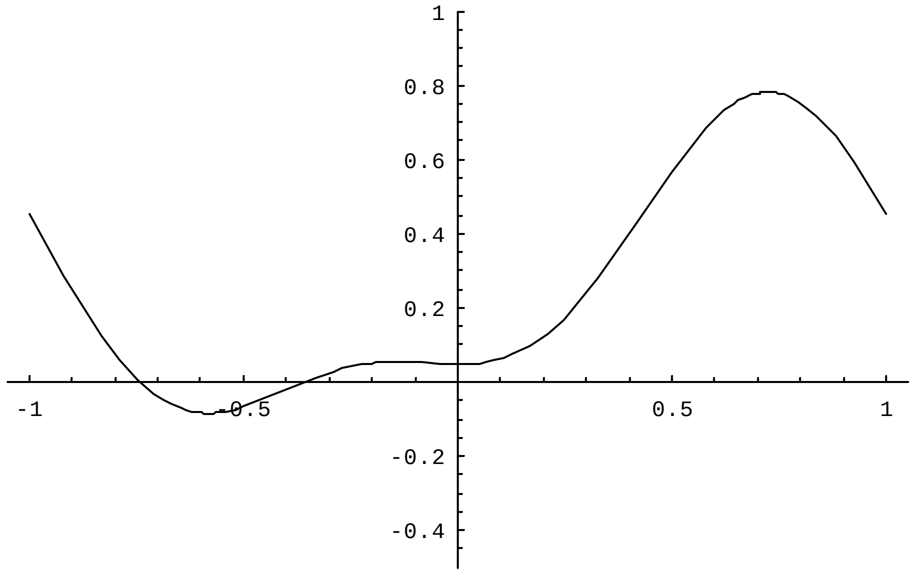
FIGGURE 7. : Legendre polynomial $y_6 = \sum_{i=0}^{6} c_i P_i$



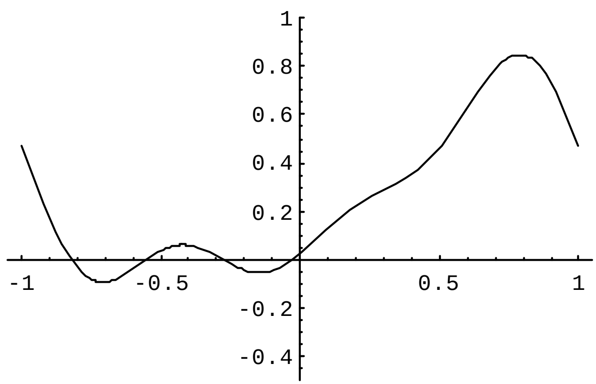
FIGGURE 8. : Fourier series $y_0 = a_0$



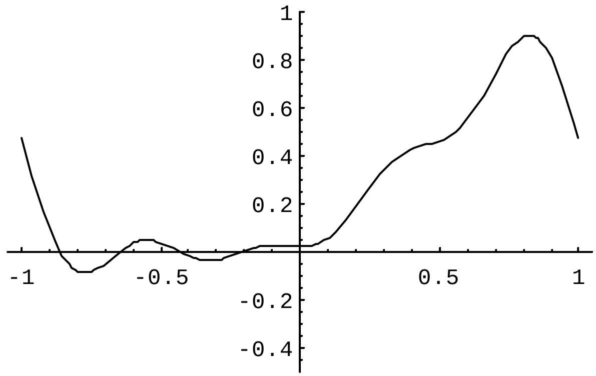
FIGGURE 9. : Fourier series $y_1 = a_0 + \hat{a}_{n=1}^{1}$ @a_n Cos HnpxL + b_n Sin HnpxLD



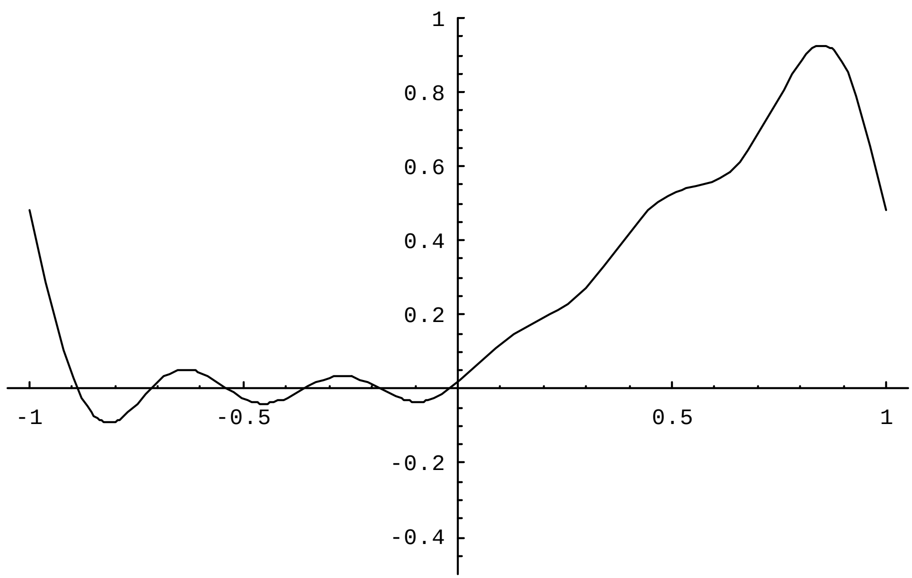
FIGGURE 10. : Fourier series $y_2 = a_0 + \sum_{n=1}^{2} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right]$



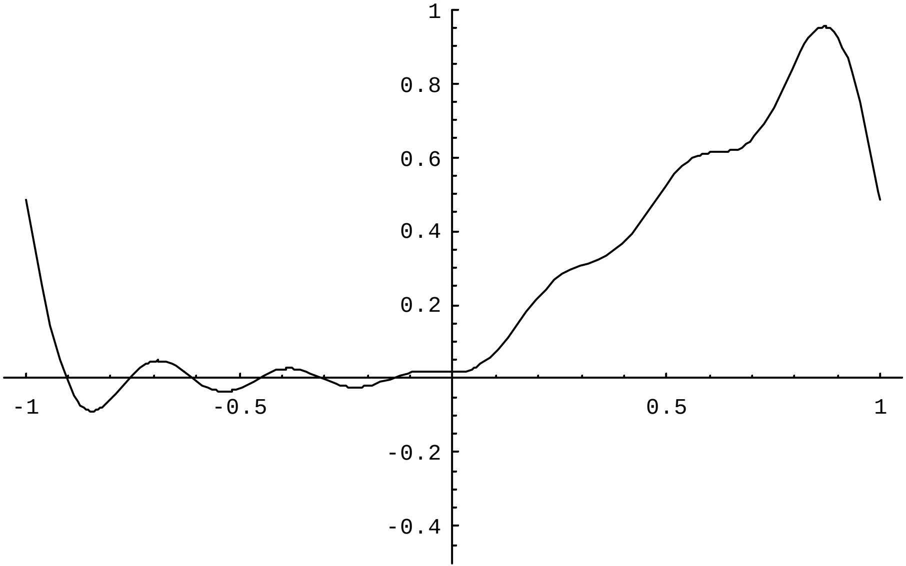
FIGGURE 11. : Fourier series $y_3 = a_0 + \sum_{n=1}^{3} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right]$



FIGGURE 12. : Fourier series $y_4 = a_0 + \overset{4}{\underset{n=1}{\hat{a}}} \; @a_n \text{ Cos HnpxL} + b_n \text{ Sin HnpxLD}$



FIGGURE 13. : Fourier series $y_5 = a_0 + \overset{5}{\underset{n=1}{\hat{a}}} \; @a_n \text{ Cos HnpxL} + b_n \text{ Sin HnpxLD}$



FIGGURE 14. : Fourier series $y_6 = a_0 + \hat{a}_{n=1}^{6}$ @a_n Cos HnpxL + b_n Sin HnpxLD