參變數法及降階法

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Statement of problem

Given two solutions y_1 and y_2 satisfy

$$a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0$$

Find a solution $y_p(x)$ satisfy

$$a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = f(x) \tag{1}$$

Review of linear algebra:

Given

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

The solution of (x, y) is

$$x = \frac{\Delta_1}{\Delta}$$

$$y = \frac{\Delta_2}{\Delta}$$

where

$$\Delta = a_1b_2 - a_2b_1$$

$$\Delta_1 = c_1b_2 - c_2b_1$$

$$\Delta_2 = a_1c_2 - a_2c_1$$

ODE:

$$a_0(x)y_1''(x) + a_1(x)y_1'(x) + a_2(x)y_1(x) = 0 \tag{2}$$
$$a_0(x)y_2''(x) + a_1(x)y_2'(x) + a_2(x)y_2(x) = 0 \tag{3}$$

Setting

$$y_p = u_1y_1 + u_2y_2 \tag{4}$$

$$y_p' = u_1'y_1 + u_2'y_2 + u_1y_1' + u_2y_2' \tag{5}$$

To solve $y_p(x)$ is changed to solve u_1 and u_2.

Two degrees of freedom, u_1 and u_2, must be determined. By setting the first constraint,

$$u_1'y_1 + u_2'y_2 = 0 \tag{6}$$

Eq.(5) can be reduced to

$$y_p' = u_1y_1' + u_2y_2' \tag{7}$$

Differentiating x again, we have

$$y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' \tag{8}$$

Substituting Eq.(8) and (7) into Eq.(1), we have

$$a_0(u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'') + a_1(u_1y_1' + u_2y_2') \qquad + a_2(u_1y_1 + u_2y_2) = f(x) \tag{9}$$

Eq.(9) can be reformulated to

$$\begin{aligned} & u_1(a_0(x)y_1''(x) + a_1(x)y_1'(x) + a_2(x)y_1(x)) \\ + \;& u_2(a_0(x)y_2''(x) + a_1(x)y_2'(x) + a_2(x)y_2(x)) \\ + \;& a_0(u_1'y_1' + u_2'y_2') = f(x) \end{aligned} \tag{10}$$

Since y_1 and y_2 are solutions of homogeneous ODE, we have

$$u_1'y_1' + u_2'y_2' = \frac{f(x)}{a_0} \tag{11}$$

Two equations are summarized

$$y_1u_1' + y_2u_2' \;=\; 0 \tag{12}$$

$$y_1'u_1' + y_2'u_2' \;=\; \frac{f(x)}{a_0} \tag{13}$$

Solve u_1' and u_2' first, we have

$$u_1' = \frac{W_1}{W(y_1, y_2)}$$

$$u_2' = \frac{W_2}{W(y_1, y_2)}$$

where $W(y_1, y_2)$ is Wronskian determined by

$$W(y_1, y_2) = y_1y_2' - y_2y_1'$$

$$W_1 = -y_2f(x)/a_0(x)$$

$$W_2 = y_1f(x)/a_0(x)$$