參變數法的廣義性

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Since y_1 and y_2 are complementary solutions for ODE, we have

$$y_1''(x)+a(x)y_1'(x)+b(x)y_1(x) = 0 \quad (1)$$
$$y_2''(x)+a(x)y_2'(x)+b(x)y_2(x) = 0 \quad (2)$$

Eq.(1) $\times$ y_2 - Eq.(2) $\times$ y_1, we have

$$\frac{d}{dx}\{y_1y_2'-y_1'y_2\}+a(x)\{y_1y_2'-y_1'y_2\}=0$$

By setting the Wronskian

$$W(x)=W(y_1,y_2)=y_1y_2'-y_1'y_2$$

$W(x)$ satisfies the following first ODE

$$W'(x)+a(x)W(x)=0$$

The solution is

$$W(x)=ke^{-\int a(x)dx}$$

Without loss of generality, given two degrees of freedoms, u_1 and u_2, must be determined. By setting the first constraint,

$$u_1'y_1+u_2'y_2=c \quad (3)$$

we have

$$u_1'y_1'+u_2'y_2'=f(x)-a(x)\ c \quad (4)$$

where c is arbitrary constant. Two equations are summarized

$$y_1u_1'+y_2u_2' = c \quad (5)$$
$$y_1'u_1'+y_2'u_2' = f(x)-a(x)\ c \quad (6)$$

Solve u_1' and u_2' first, we have

$$u_1'=\frac{-f(x)y_2}{W(y_1,y_2)}+c(\frac{y_2'+ay_2}{W})$$

$$u_2'=\frac{f(x)y_1}{W(y_1,y_2)}-c(\frac{y_1'+ay_1}{W})$$

The two additional terms containg c are present. It is interesting to find that

$$u_1'=c(\frac{y_2'+ay_2}{W})(\frac{e^{\int a(x)dx}}{e^{\int a(x)dx}})=c(y_2e^{\int a(x)dx})'$$

$$u_2;c(\frac{y_1'+ay_1}{W})(\frac{e^{\int a(x)dx}}{e^{\int a(x)dx}})=c(y_1e^{\int a(x)dx})'$$

They can be cancelled each other.