1.

$$
\begin{aligned}
&\int_a^b u(x)\mathcal{L}\{v(x)\}dx \\
&= \int_a^b u(x)[[p(x)\frac{dv(x)}{dx}]' + r(x)v(x)]dx \\
&= \int_a^b u(x)[p(x)\frac{dv(x)}{dx}]'dx + \int_a^b u(x)r(x)v(x)dx \\
&= u(x)(p(x)\frac{dv(x)}{dx})|_{x=a}^{x=b} - \int_a^b u'(x)p(x)\frac{dv}{dx}dx + \int_a^b u(x)r(x)v(x)dx \\
&= u(x)(p(x)\frac{dv(x)}{dx})|_{x=a}^{x=b} - [u'(x)p(x)v(x)|_{x=a}^{x=b} - \int_a^b [u'(x)p(x)]'v(x)dx] + \int_a^b u(x)r(x)v(x)dx \\
&= u(x)(p(x)\frac{dv(x)}{dx})|_{x=a}^{x=b} - \frac{du(x)}{dx}p(x)v(x)|_{x=a}^{x=b} + \int_a^b [[p(x)\frac{du(x)}{dx}]' + r(x)u(x)]v(x)dx \\
&= u(x)(p(x)\frac{dv(x)}{dx})|_{x=a}^{x=b} - \frac{du(x)}{dx}p(x)v(x)|_{x=a}^{x=b} + \int_a^b v(x)\mathcal{L}^{\dagger}\{u(x)\}dx \\
&\therefore \mathcal{L}^{\dagger} = \mathcal{L} = (p(x)\frac{d}{dx})' + r(x) \\
&J(u,v) = [u(x)(p(x)\frac{dv(x)}{dx}) - \frac{du(x)}{dx}(p(x)v(x))]|_{x=a}^{x=b}
\end{aligned}
$$