1. Define 操作元= $\frac{d^2 y_n(x)}{dx^2} + \frac{1}{4} y_n(x) = 0$

求 $\frac{d^2 y_n(x)}{dx^2} + \frac{1}{4} y_n(x) = f(x) = \frac{x}{2}$ 的 close-form，BoundaryCondition：

$y(0) = Y(p) = 0$

解：(1)建立資料庫 ， l_n, y_n

$$\frac{d^2 y_n(x)}{dx^2} + \frac{1}{4} y_n(x) = l_n y_n(x)$$

$$\frac{d^2 y_n(x)}{dx^2} + (\frac{1}{4} - l_n) y_n(x) = 0$$

$\Rightarrow \frac{1}{4} - l = n^2$ 時有非無聊解，

$y_n(x) = A\sin(nx) + Bcon(nx)$ 代入 Boundary Condition

$$\Rightarrow y_n(x) = A\sin(nx) \qquad l_n = \frac{1}{4} - n^2 \qquad n = 0,1,2,3,4.........$$

The normalization condition further requires

$$\int_0^p A^2 \sin^2(nx) dx = 1 \quad \Rightarrow \quad A = \sqrt{\frac{2}{p}}$$

From Green’s function

$$G(x,s) = \sum_{n=0}^{\infty} \frac{1}{l_n} y_n(x) \cdot y_n(s)$$

$$= \frac{2}{p} \sum_{n=0}^{\infty} \frac{\sin(nx) \cdot \sin(ns)}{\frac{1}{4} - n^2}$$

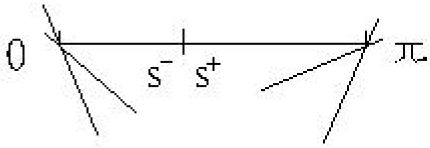
$$\Rightarrow y(x) = \int_0^p G(x,s) f(s) ds$$

$$= \int_0^p \left(\frac{2}{p} \sum_{n=0}^{\infty} \frac{\sin(nx)\sin(ns)}{\frac{1}{4} - n^2} \right) \cdot \frac{s}{2} ds = \frac{1}{p} \sum_{n=0}^{\infty} \frac{\sin(nx)}{\frac{1}{4} - n^2} \int_0^p s \cdot \sin(ns) ds$$

$$= \frac{1}{p} \sum_{n=0}^{\infty} \frac{\sin(nx)}{\frac{1}{4} - n^2} \left[\frac{p}{n} (-1)^{n+1} \right]$$

(上式的積分項在 n=0 時為零)

$$\Rightarrow y_n(x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n(\frac{1}{4}-n^2)}$$



如圖，左右兩端都須符合 $\frac{d^2y_n(x)}{dx^2}+\frac{1}{4}y_n(x)=0$

在 x=s 處左右兩端分開計算得到 y_1, y_2

$$\Rightarrow \begin{matrix} y_1=A\sin\frac{1}{2}x \quad y(0)=0 \\ y_2=B\cos\frac{1}{2}x \quad y(\pi)=0 \end{matrix}$$

$\langle 1\rangle$ **此桿子在 x=s 處不斷裂**

$$\Rightarrow y_1=y_2 \Rightarrow A\sin\frac{1}{2}x=B\cos\frac{1}{2}x \text{ ....}\textbf{..(1)}$$

$$\langle 2\rangle\int_{S^-}^{S^+}\left(\frac{d^2y}{dx^2}+\frac{1}{4}y\right)dx=\int_{S^-}^{S^+}\delta(x,s)dx$$

$$\Rightarrow \frac{dy}{dx}\Big|_{s^-}^{s^+}+\frac{1}{4}\int_{S^-}^{S^+}(y)dx=1 \qquad \Rightarrow \frac{dy}{dx}\Big|_{s^-}^{s^+}=1$$

(即在 x=s 處左右一點點處的斜率差值為 1)

$$\Rightarrow -\frac{1}{2}B\sin\frac{1}{2}x-\frac{1}{2}\cos\frac{1}{2}x=1 \text{ ....}\textbf{(2)}$$

將(1),(2)兩式解聯立解 $\Rightarrow A=-2\cos\frac{1}{2}x \quad , \quad B=-2\sin\frac{1}{2}x$

$$\Rightarrow G(x,s)=\left\{-2\cos\frac{1}{2}x\cdot\sin\frac{1}{2}s \quad s\prec x^- \quad , \; -2\sin\frac{1}{2}x\cdot\cos\frac{1}{2}s \quad s\succ x^+\right\}$$

代入 $y(x)=\int_0^{\pi}G(x,s)f(s)ds$

$$=\int_0^{x^-}-2\cos\frac{1}{2}x\cdot\sin\frac{1}{2}s\cdot\frac{s}{2}ds+\int_{x^+}^{\pi}-2\sin\frac{1}{2}x\cdot\cos\frac{1}{2}s\cdot\frac{s}{2}ds$$

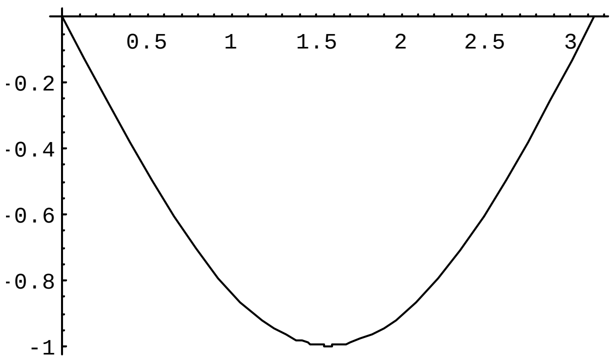
$=2x-2\pi\sin(x/2)$ **此解與用解 ODE 的方式所得的解(exact solve)相同。**

```
t = 100
```

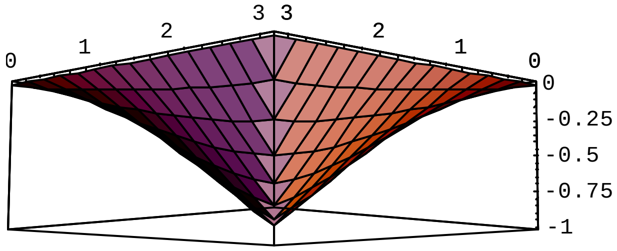
```
                 t
             2        Sin@n * xD * Sin@n * sD
G@x_, s_D := ê * ã   ê;
             p               1
                n=1          ê - n^2
                             4
```

```
Plot@G@x, xD, 8x, 0, p<D
Plot3D@G@x, sD, 8x, 0, p<, 8s, 0, p<, ViewPoint ® 83, 3, -0.5<D
```

100

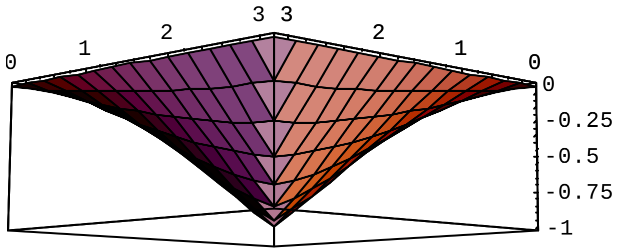


… Graphics …



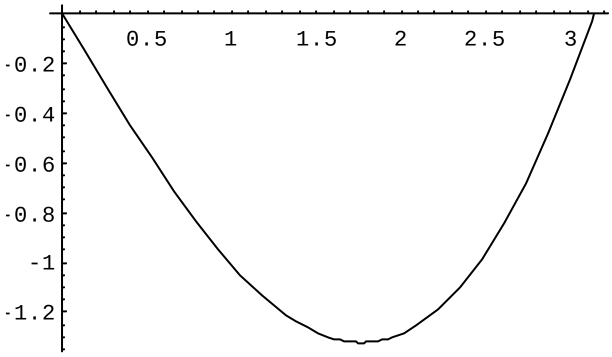
… SurfaceGraphics …

```
                          s          x
G@x_, s_D := -2 CosAê E * SinAê E •; 0 £ x £ s;
                          2          2
                          s          x
G@x_, s_D := -2 SinAê E * CosAê E •; s < x £ p;
                          2          2
Plot3D@G@x, sD, 8x, 0, p<, 8s, 0, p<, ViewPoint ® 83, 3, -0.5<D
```



… SurfaceGraphics …

```
Y@x_D := 2 * x - 2 p * Sin@x • 2D
Plot@Y@xD, 8x, 0, p<D
```



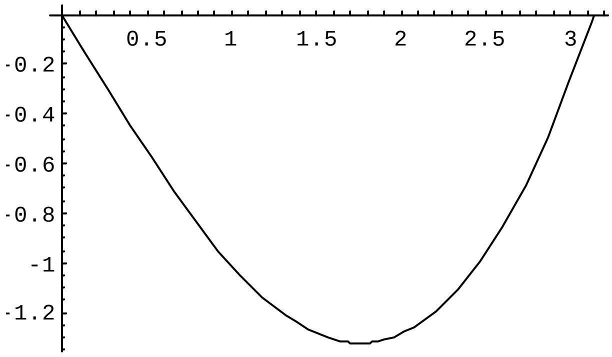
… Graphics …

… Graphics …

```
t = 10
```

```
            t
           H-1L^(n+1) * Sin@n * xD
Y@x_D :=  ã  ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢ ;
           n I 1/4 - n^2 M
          n=1
```

```
Plot@Y@xD, 8x, 0, p<D
```



… Graphics …