系級:______________ 姓名:______________ 學號:______________

國立臺灣海洋大學河海工程學系 2001 工程數學 (三) 期末作業小考解答

1. Solve the Green's function in terms of closed-form solution for

$$\frac{d^2y(x)}{dx^2}+\pi^2 y(x)=\delta(x-s)$$

subject to $y(0)=y(1)$ and $y'(0)=y'(1)$.

2. Solve the particular solution for

$$\frac{d^2y(x)}{dx^2}+\pi^2 y(x)=cos(\pi x).$$

Sol.

(一). Method of variation of parameters.

$$y_n=a\cos(\pi x)+b\sin(\pi x),\quad y_p=u_1\cos(\pi x)+u_2\sin(\pi x)$$

$$u_1'\cos(\pi x)+u_2'\sin(\pi x)=0$$

$$-u_1'\sin(\pi x)+u_2'\cos(\pi x)=\cos(\pi x)$$

$$u_1'=\frac{-\sin(\pi x)\cos(\pi x)}{\pi},\quad u_2'=\frac{\cos^2(\pi x)}{\pi}$$

$$u_1=\frac{\cos(2\pi x)}{4\pi^2},\quad u_2=\frac{x}{2\pi}+\frac{\sin(2\pi x)}{4\pi^2}$$

$$\therefore y_p=a\cos(\pi x)+b\sin(\pi x)+u_1\cos(\pi x)+u_2\sin(\pi x)$$

$$=a\cos(\pi x)+b\sin(\pi x)+\frac{x\sin(\pi x)}{2\pi}+\frac{\cos(\pi x)}{4\pi^2}$$

代入 $y(0)=y(1),\ \ y'(0)=y'(1),\quad a=\frac{-1}{4\pi^2},\ \ b=\frac{-1}{4\pi^2}$

$$y_p=\frac{x\sin(\pi x)}{2\pi}-\frac{\sin(\pi x)}{4\pi}$$

(二). 分段法 closed-form solution

$$y_1=a\cos(\pi x)+b\sin(\pi x)$$

$$y_2=c\cos(\pi x)+d\sin(\pi x)$$

$$y_1'=-a\pi\sin(\pi x)+b\pi\cos(\pi x)$$

$$y_2'=-c\pi\sin(\pi x)+d\pi\cos(\pi x)$$

代入條件: $y_1(0)=y_2(1),\ \ y_1'(0)=y_2'(1),\ \ y'(x)|_{x=s^-}^{x=s^+}=1,\ \ y_1(s^-)=y_2(s^+)$

$$a=\frac{\sin(\pi s)}{2\pi},\ \ b=-\frac{\cos(\pi s)}{2\pi},\ \ c=-\frac{\sin(\pi s)}{2\pi},\ \ d=\frac{\cos(\pi s)}{2\pi}$$

$$G(x,s)=\begin{cases}\frac{\sin(\pi s)}{2\pi}\cos(\pi x)-\frac{\cos(\pi s)}{2\pi}\sin(\pi x), & x<s\\ -\frac{\sin(\pi s)}{2\pi}\cos(\pi x)+\frac{\cos(\pi s)}{2\pi}\sin(\pi x), & x>s\end{cases}$$

$$y_p = \int_0^1 G(x,s)f(s)ds$$

$$= \int_0^x y_2 \cos(\pi s)ds + \int_x^1 y_1 \cos(\pi s)ds$$

$$= \frac{x\sin(\pi x)}{2\pi} - \frac{\sin(\pi x)}{4\pi}$$

(三). series-form solution

$$y''(x) + \pi^2 y(x) = \cos(\pi x), \quad y(0) = y(1), \quad y'(0) = y'(1)$$

$$y_p = \sum C_n y_n(x), \quad \mathcal{L}(y_n) = y_n''(x) + \pi^2 y_n(x) = \lambda_n y_n(x)$$

$$y_n''(x) + (\pi^2 - \lambda_n)y_n(x) = 0$$

(1). $\pi^2 - \lambda_n = 0 \quad (\times)$

(2). $\pi^2 - \lambda_n < 0, \quad \pi^2 - \lambda_n = -\omega^2, \quad y_n'' - \omega^2 y_n = 0, \quad y_n = c_1 e^{\omega x} + c_2 e^{-\omega x}$

代入 $y(0) = y(1), \quad y'(0) = y'(1), \quad \omega = 0 \quad (\times)$

(3). $\pi^2 - \lambda_n > 0$

$$\pi^2 - \lambda_n = \omega^2, \quad y_n'' + \omega^2 y_n = 0, \quad y_n = a\cos(\omega x) + b\sin(\omega x)$$

代入 $y(0) = y(1), \quad y'(0) = y'(1)$

$$a(\cos(\omega) - 1) + b\sin(\omega) = 0$$

$$-a\sin(\omega) + b(\cos(\omega) - 1) = 0, \quad \because a, b \neq 0$$

$$\begin{vmatrix} \cos(\omega) - 1 & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) - 1 \end{vmatrix} = 0, \quad \omega = 2n\pi, \quad n = 0, 1, 2, ...$$

$$\lambda_n = \pi^2(1 - 4n^2), \quad y_n = a\cos(2n\pi x) + b\sin(2n\pi x), \quad n = 0, 1, 2, ...$$

$$G(x,s) = \sum_{n=0}^{\infty} \frac{\sin(2n\pi x)\sin(2n\pi s) + \cos(2n\pi x)\cos(2n\pi s)}{\pi^2(1-4n^2)}$$

$$y_p = \sum_{n=0}^{\infty} C_n y_n = \sum_{n=0}^{\infty} \frac{d_n}{\lambda_n} y_n$$

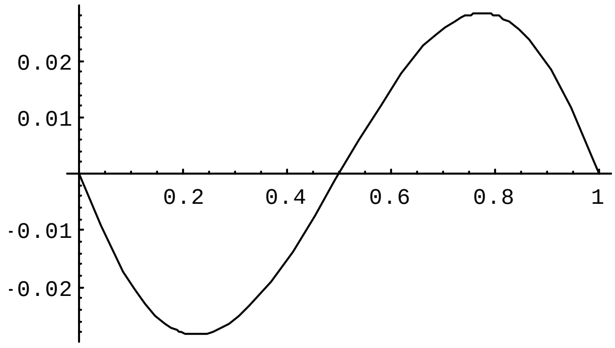
$$d_n = \frac{\int_0^1 y_n(x)f(x)dx}{\|y_n(x)\|^2} = \frac{\frac{4nb}{(4n^2-1)\pi}}{\frac{1}{2}(a^2+b^2)} = \frac{8nb}{\pi(4n^2-1)(a^2+b^2)}$$

令$\|y_n\|^2 = \frac{1}{2}(a^2+b^2) = 1$, use $a = 0, \quad b = \sqrt{2}$

$$y_p = \sum_{n=0}^{\infty} \left(\frac{4n\sqrt{2}}{(4n^2-1)\pi} \cdot \frac{1}{\pi^2(1-4n^2)}\right)\sqrt{2}\sin(2n\pi x)$$

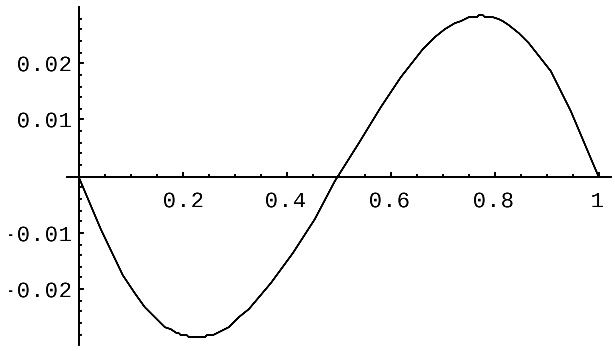
$$= \sum_{n=0}^{\infty} \frac{-8n\sin(2n\pi x)}{\pi^3(4n^2-1)^2}$$

```
      x                1
PlotA€€€€€€ Sin@p xD - €€€€€€ Sin@p xD, 8x, 0, 1<EH* closed-form solution *L
      2 p              4 p
```



… Graphics …

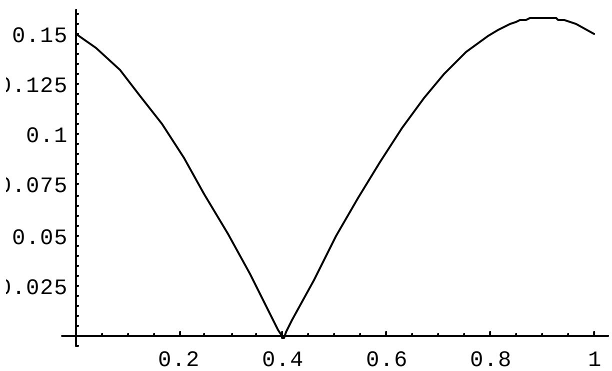
```
        100
         ‚  i -8 n HSin@2 n p xDL y
PlotA   ã   j €€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€ z, 8x, 0, 1<E H* serier solution *L
            k  p3 H4 n2 - 1L2     {
        n=1
```



… Graphics …

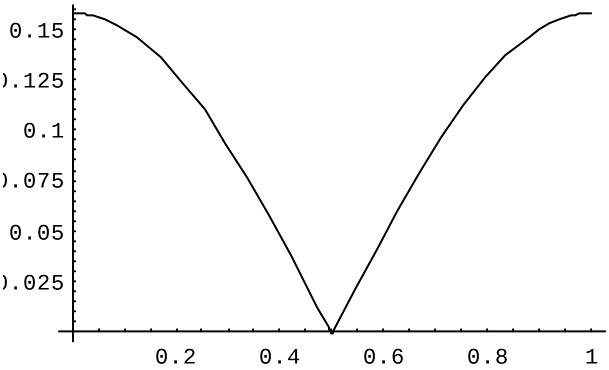
```
           1000
            ‚  i 2 Sin@2 n p xD Sin@2 n p 0.4D + 2 Cos@2 n p xD Cos@2 n p 0.4D y
PlotA0.1 +  â   j €€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€ z,
               k                        p2 - 4 n2 p2                          {
           n=1

 8x, 0, 1<, PlotRange ® AllEH* Green function G Hx,0.4L *L
```



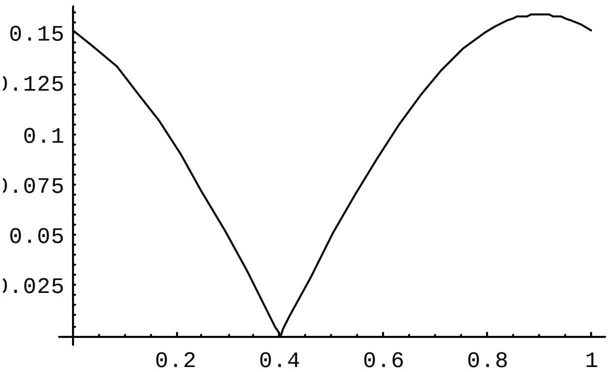
… Graphics …

```
          1000 i        1
PlotA0.1 +  â  j‡‡‡‡‡‡‡‡‡‡‡‡‡‡‡‡‡‡‡‡‡ H2 Sin@2 n p xD Sin@2 n p 0.5D + 2 Cos@2 n p xD Cos@2 n p 0.5DLy,
           n=1 k p2 - 4 n2 p2                                                                 {
 8x, 0, 1<, PlotRange ® AllEH* G Hx,0.5L *L
```



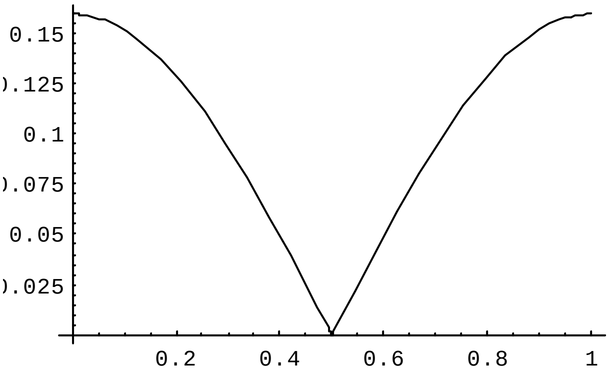
… Graphics …

```
         i Sin@p 0.4D                 Cos@p 0.4D              y
g@x_D := j‡‡‡‡‡‡‡‡‡‡‡‡‡‡‡ Cos@p xD - ‡‡‡‡‡‡‡‡‡‡‡‡‡‡‡ Sin@p xDz •; 0 < x < 0.4
         k     2 p                       2 p                  {
         i  Sin@p 0.4D                 Cos@p 0.4D              y
g@x_D := j- ‡‡‡‡‡‡‡‡‡‡‡‡‡‡‡ Cos@p xD + ‡‡‡‡‡‡‡‡‡‡‡‡‡‡‡ Sin@p xDz •; 0.4 < x < 1
         k      2 p                       2 p                  {
Plot@g@xD, 8x, 0, 1<DH* G Hx,0.4L *L
```



… Graphics …

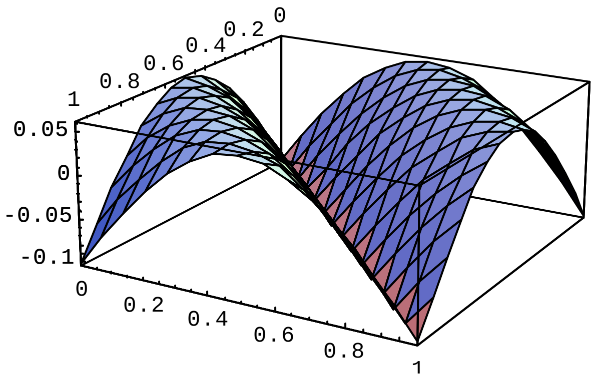
```
          i Sin@p 0.5D                    Cos@p 0.5D                 y
f@x_D := j ((((((((((((((((( Cos@p xD - ((((((((((((((((( Sin@p xD z •; 0 < x < 0.5
          k      2 p                          2 p                    {
          i   Sin@p 0.5D                    Cos@p 0.5D                y
f@x_D := j - ((((((((((((((((( Cos@p xD + ((((((((((((((((( Sin@p xD z •; 0.5 < x < 1
          k        2 p                          2 p                   {
Plot@f@xD, 8x, 0, 1<DH* G Hx,0.5L *L
```



… Graphics …

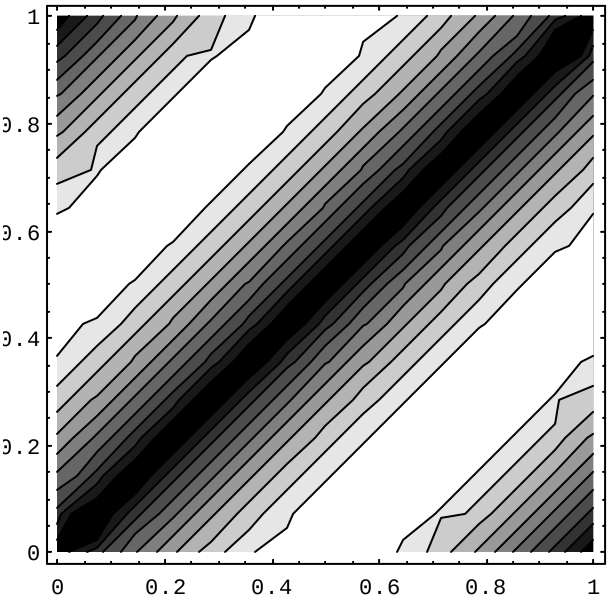
```
         1000
          â   i 2 Sin@2 n p xD Sin@2 n p sD + 2 Cos@2 n p xD Cos@2 n p sD y
Plot3DA      j ((((((((((((((((((((((((((((((((((((((((((((((((((((((((((((( z, 8x, 0, 1<,
         n=1  k                       p^2 - 4 n^2 p^2                      {

 8s, 0, 1<, ViewPoint -> 84.000, 2.320, 1.580<EH* G Hx,sL, 0<x,s<1 *L
```



… SurfaceGraphics …

```
        1000  i 2 Sin@2 n p xD Sin@2 n p sD + 2 Cos@2 n p xD Cos@2 n p sD y
ContourPlotA â  j ------------------------------------------------------------ z,
        n=1   k                    p² - 4 n² p²                               {
 8x, 0, 1<, 8s, 0, 1<EH* G Hx,sL, 0<x,s<1 *L
```



… ContourGraphics …