國立臺灣海洋大學河海工程學系 2001 工程數學 (三) 期中考參考解答

1. (a) Express the $f(x) = x^2 - 1$ in terms of Lengendre polynomials

$$P_0(x) = 1,$$
$$P_1(x) = x,$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1),$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x),$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

Find the particular solution, $y_p(x)$, for (25 %)

$$(1-x^2)y_p''(x) - 2xy_p'(x) = x^2 - 1$$

Ans: $f(x) = \frac{2}{3}P_2(x) - \frac{2}{3}$.
Ans: $y_p(x) = \frac{-1}{9}P_2(x) + \frac{1}{3}ln \mid (x+1)(x-1) \mid$

2. By changing the variable of $\bar{x} = \sqrt{x}$, tranform the ODE

$$4xy''(x) + 2y'(x) + y(x) = 0$$

to ODE for $\bar{y}(\bar{x})$ and solve $\bar{y}(\bar{x})$ and $y(x)$, where $y(x) = \bar{y}(\bar{x})$. (25 %)
Ans:

$$\frac{dy(x)}{dx} = \frac{1}{2\bar{x}}\frac{d\bar{y}(\bar{x})}{d\bar{x}}$$
$$\frac{d^2y(x)}{dx^2} = \frac{1}{2\bar{x}}\{\frac{1}{2\bar{x}}\frac{d^2\bar{y}(\bar{x})}{d\bar{x}^2} - \frac{1}{2\bar{x}^2}\frac{d\bar{y}(\bar{x})}{d\bar{x}}\}$$
$$\frac{d^2\bar{y}(\bar{x})}{d\bar{x}^2} + \bar{y}(\bar{x}) = 0$$

3. Prove that the eigenvalues of Hermitian matrice are real. (25 %)
Ans:

$$\mathbf{x}\cdot(A\mathbf{x}) = \mathbf{x}\cdot(\lambda\mathbf{x}) = \mathbf{x}^T(\bar{\lambda}\bar{\mathbf{x}}) = \bar{\lambda}(\mathbf{x}\cdot\mathbf{x})$$
$$A\mathbf{x}\cdot\mathbf{x} = \lambda\mathbf{x}\cdot\mathbf{x} = \lambda(\mathbf{x}\cdot\mathbf{x})$$
$$\mathbf{x}\cdot(A\mathbf{x}) = \bar{A}^T\mathbf{x}\cdot\mathbf{x}; A\mathbf{x}\cdot\mathbf{x} = \mathbf{x}\cdot(\bar{A}^T\mathbf{x})$$

Since A is Hermitian matrice, we have

$$(\lambda - \bar{\lambda})(\mathbf{x}\cdot\mathbf{x}) = 0$$

Since the norm of $\mathbf{x}$ is greater than zero, we have the real eigenvalues for λ.

4. Given $[A]\mathbf{x} = \mathbf{b}$ where

$$[A] = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of $[A]$. Using the eigenvector expansion, solve $\mathbf{x}$. (25 %)
Ans: eigenvalues: $-1, -2$.
Ans: eigenvectors: $\{1,1\}^T, \{2,3\}^T$
Ans: $\mathbf{x}^T = \{-2, -2.5\}$.