國立臺灣海洋大學河海工程學系 2002 工程數學 (三) 期末考參考解答 (ma3fins.ctx)

1. Explain the following items. (15 %)
(1). Symmetric matrix, $A = A^T$, (2). Hermitian matrix, $A = \bar{A}^T$, (3). Hermitian operator, $\mathcal{L} = \mathcal{L}^\dagger$ (4). Green's function $\mathcal{L}\{G(x,s)\} = \delta(x-s)$ with boundary condition and (5). Fundamental solution (free-space Green's function) $\mathcal{L}\{G(x,s)\} = \delta(x-s)$ without boundary condition

2. Determine the eigenvalues $\lambda_n = -(n+\frac{1}{2})^2\pi^2, n = 0,1,2,3...$ and eigenfunctions $y_n(x) = sin((n+\frac{1}{2})\pi x)$ for $\frac{d^2 y_n(x)}{dx^2} = \lambda_n y_n(x)$ subject to $y_n(0) = 0$ and $y'(1) = 0$. (10 %) Solve the Green's function $y(x) = G(x,s) = -s, x > s, -x, x < s$ in terms of closed form and series form for

$$\frac{d^2 y(x)}{dx^2} = \delta(x-s)$$

subject to $y(0) = 0$ and $y'(1) = 0$. (10 %) Solve the particular solution in terms of closed form and series form for

$$\frac{d^2 y(x)}{dx^2} = cos(x)$$

subject to $y(0) = 0$ and $y'(1) = 0$. (10 %) $y(x) = -cos(x) + 1 - sin(1)x$.

3. Change the two ODEs to the Sturm-Liouville forms of $(py')' + qy = -\lambda\rho y$. (20 %)

$$y''(x) - 2xy'(x) + 2\alpha y(x) = 0 \;\rightarrow\; e^{-x^2} \;\; and \;\; (1-x^2)y''(x) - xy'(x) + n^2 y(x) = 0 \;\rightarrow\; \frac{1}{1-x^2}$$

Please detemine the functions of p, q, ρ and the eigenvalue of λ.

4. Find the solution

$$(1-x^2)y''(x) - 2xy'(x) + 14y(x) = 5x^3$$

where the Lengendre polynomials are $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2-1)$ and $P_3(x) = \frac{1}{2}(5x^3-3x)$. First determine c_n such that $5x^3 = \sum_{n=0}^{3} c_n P_n(x)$. (10 %) Then solve the particular solution using eigenfunction expansion. (10 %)
$5x^3 = 2P_3(x) + 3P_1(x)$ and $y(x) = 5(2x^3 - x)/4$.

5. By substituting $x = e^t$, find the normalized eigenfunction $y_n(x)$ (10 %) and the eigenvalues λ_n (10 %) for the operator $\mathcal{L}$ defined by

$$\mathcal{L} = x^2 y'' + 2xy' + \frac{1}{4}y, \quad \text{subject to} \quad y(1) = y(e) = 0$$

Find the solution for $\mathcal{L}\{y(x)\} = \frac{1}{\sqrt{x}}$ using $y(x) = \sum a_n y_n(x)$. (10 %)
$y_n(x) = \sqrt{2}x^{-1/2} sin(n\pi\, ln(x))$ with respect to $\lambda_n = -n^2\pi^2$
$a_n = -(n\pi)^{-2}\int_1 e\sqrt{2}x^{-1} sin(n\pi\, ln(x)) = -\sqrt{8}(n\pi)^{-3}$

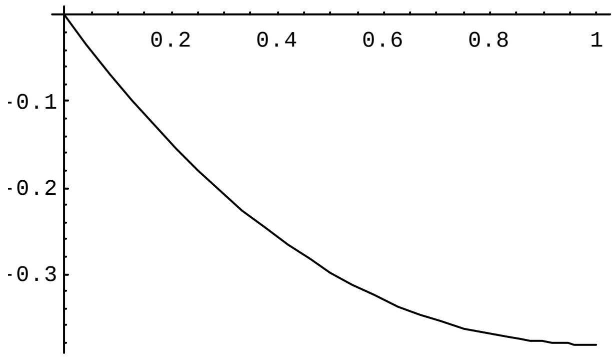
6. Solve the Green's function in terms of closed form for

$$\frac{d^2 y(x)}{dx^2} + \pi^2 y(x) = \delta(x-s)$$

subject to $y(0) = y(1)$ and $y'(0) = y'(1)$. (10 %) Solve the particular solution for

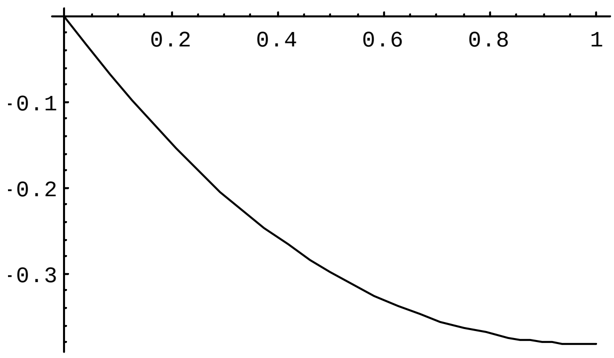
$$\frac{d^2 y(x)}{dx^2} + \pi^2 y(x) = cos(\pi x).\ (10\ \%)$$

```
Plot@1 - Cos@xD - x Sin@1D, 8x, 0, 1<D
```



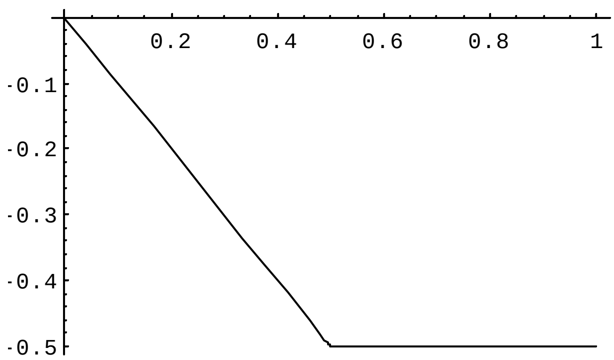
… Graphics …

```
          1000  i   i CosA 2n-1 p+1E-1     CosA 2n-1 p-1E-1 y         y
                  4 j ------------------ + ------------------ z         z
                    k  I 2n-1 p+1M           2n-1 p-1         {         z
PlotA ã   ‚     ------------------------------------------------- SinA H2 n - 1L p x E z, 8x, 0, 1<, PlotRange ® AllE
                               H2 n - 1L^2 p^2                         ---------------- z
          n=1                                                                 2         z
                k                                                                       {
```



… Graphics …

```
f@x_D := -x •; 0 < x < 0.5
f@x_D := -0.5 •; 0.5 < x < 1
Plot@f@xD, 8x, 0, 1<D
```



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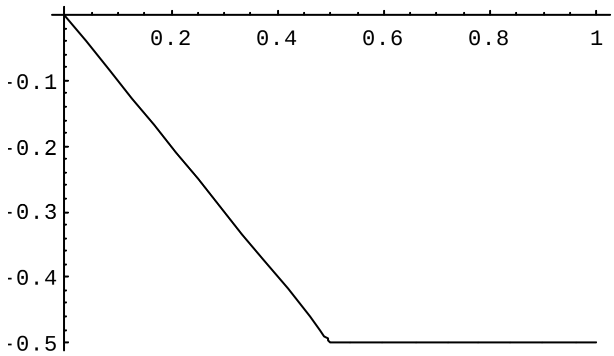
```
          100  i -8 SinA 2n-1 p xE SinA 2n-1 p 0.5E y
                          2              2
PlotA ã  j                                       z, 8x, 0, 1<, PlotRange ® AllE
                       H2 n - 1L^2 p^2
          n=1  k                                   {
```



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