

工程數學 (四) - 偏微分方程

I. By using the auxilliary system $U(x, s; t, \tau)$, which is a solution of

$$\frac{\partial^2 U(x, s; t, \tau)}{\partial t^2} - c^2 \frac{\partial^2 U(x, s; t, \tau)}{\partial x^2} = \delta(x - s)\delta(t - \tau), -\infty < x < \infty, t > 0$$

with initial conditions

$$\lim_{t \rightarrow \tau} U(x, s; t, \tau) = 0$$

$$\lim_{t \rightarrow \tau} \dot{U}(x, s; t, \tau) = 0$$

and no boundary condition since $-\infty < x < \infty$,

Solve the problem

$$u_{tt} = c^2 u_{xx}, \quad \text{for } 0 < x < 1, \quad t > 0$$

with initial conditions

$$u(x, 0) = 0, \quad u_t(x, 0) = 0$$

and boundary conditions

$$u(0, t) = a(t), \quad u(1, t) = b(t)$$

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \int_0^t \int_B \mathbf{U}(\mathbf{s}, \mathbf{x}; t, \tau) \cdot \mathbf{t}(\mathbf{s}, \tau) dB(\mathbf{s}) d\tau - \int_0^t \int_B \mathbf{T}(\mathbf{s}, \mathbf{x}; t, \tau) \cdot \mathbf{u}(\mathbf{s}, \tau) dB(\mathbf{s}) d\tau, \\ &= - \int_0^t \mathbf{T}(\mathbf{s}, \mathbf{x}; t, \tau) \cdot \mathbf{u}(\mathbf{s}, \tau) \Big|_{s=0}^{s=l} d\tau, \end{aligned} \quad (1)$$

$$\begin{aligned} \mathbf{t}(\mathbf{x}, t) &= \int_0^t \int_B \mathbf{L}(\mathbf{s}, \mathbf{x}; t, \tau) \cdot \mathbf{t}(\mathbf{s}, \tau) dB(\mathbf{s}) d\tau - \int_0^t \int_B \mathbf{M}(\mathbf{s}, \mathbf{x}; t, \tau) \cdot \mathbf{u}(\mathbf{s}, \tau) dB(\mathbf{s}) d\tau \\ &= - \int_0^t \mathbf{M}(\mathbf{s}, \mathbf{x}; t, \tau) \cdot \mathbf{u}(\mathbf{s}, \tau) \Big|_{s=0}^{s=l} d\tau, \end{aligned} \quad (2)$$

where $\mathbf{t}(\mathbf{x}, t) = \mathbf{u}'(\mathbf{x}, t)$ and $\mathbf{U}(\mathbf{s}, \mathbf{x}; t, \tau)$, $\mathbf{T}(\mathbf{s}, \mathbf{x}; t, \tau)$, $\mathbf{L}(\mathbf{s}, \mathbf{x}; t, \tau)$ and $\mathbf{M}(\mathbf{s}, \mathbf{x}; t, \tau)$ are four kernel functions which can be expressed in terms of series solution

$$\mathbf{U}(\mathbf{s}, \mathbf{x}; t, \tau) = \sum_{m=1}^{\infty} \frac{1}{N_m \omega_m} \sin(\omega_m(t - \tau)) \mathbf{u}_m(\mathbf{x}) \mathbf{u}_m(\mathbf{s}), \quad (3)$$

$$\mathbf{T}(\mathbf{s}, \mathbf{x}; t, \tau) = \sum_{m=1}^{\infty} \frac{1}{N_m \omega_m} \sin(\omega_m(t - \tau)) \mathbf{u}_m(\mathbf{x}) \mathbf{t}_m(\mathbf{s}), \quad (4)$$

$$\mathbf{L}(\mathbf{s}, \mathbf{x}; t, \tau) = \sum_{m=1}^{\infty} \frac{1}{N_m \omega_m} \sin(\omega_m(t - \tau)) \mathbf{t}_m(\mathbf{x}) \mathbf{u}_m(\mathbf{s}), \quad (5)$$

$$\mathbf{M}(\mathbf{s}, \mathbf{x}; t, \tau) = \sum_{m=1}^{\infty} \frac{1}{N_m \omega_m} \sin(\omega_m(t - \tau)) \mathbf{t}_m(\mathbf{x}) \mathbf{t}_m(\mathbf{s}), \quad (6)$$

in which $\mathbf{u}_m(\mathbf{x}) = \sin(m\pi x/l)$, $\mathbf{t}_m(\mathbf{x}) = m\pi \cos(m\pi x/l)/l$, $N_m = l/2$, $\omega_m = m\pi c/l$, D is $(0, l)$ and B is 0 , and l .

$$U(x, s; t, \tau) = 0, \quad \text{if } t < \tau$$

$$U(s, x; \tau, t) = 0, \quad \text{if } \tau < t$$

$$U(x, s; t, \tau) = U(s, x; t, \tau)$$

$$U(x, s; t, \tau) \neq U(x, s; \tau, t)$$

$$U(x, s; t, \tau) \neq U(s, x; \tau, t)$$

However, both $U(x, s; t, \tau)$ and $U(s, x; \tau, t)$ satisfy the governing equation

$$\frac{\partial^2 U(x, s; t, \tau)}{\partial t^2} - c^2 \frac{\partial^2 U(x, s; t, \tau)}{\partial x^2} = \delta(x - s)\delta(t - \tau), -\infty < x < \infty, t > 0$$

$$\frac{\partial^2 U(s, x; \tau, t)}{\partial t^2} - c^2 \frac{\partial^2 U(s, x; \tau, t)}{\partial x^2} = \delta(x - s)\delta(t - \tau), -\infty < x < \infty, t > 0$$

$$\delta(x-s) = \sum_{m=1}^{\infty} \frac{2}{l} \sin(m\pi x/l) \sin(m\pi s/l)$$

Exact form:

$$U(x, s; t, \tau) = \frac{1}{2c} H(x-s+c(t-\tau)) - \frac{1}{2c} H(x-s-c(t-\tau))$$

Series form:

$$\mathbf{U}(\mathbf{s}, \mathbf{x}; t, \tau) = \sum_{m=1}^{\infty} \frac{1}{N_m \omega_m} \sin(\omega_m(t-\tau)) \sin(m\pi x/l) \sin(m\pi s/l)$$

Choosing $u = u(x, t)$ and $v(x, t) = U(s, x; \tau, t)$,

$$\begin{aligned} u(s, \tau) &= \int_0^{\tau^+} \int_V u(x, t) (\delta(x-s) \delta(t-\tau)) dV(x) dt \\ &= \int_V [u(x, t) \dot{U}(s, x; \tau, t) - U(s, x; \tau, t) \dot{u}(x, t)] \Big|_{t=0}^{\tau^+} dV(x) \\ &\quad - c^2 \int_0^{\tau^+} \int_S [u(x, t) \frac{\partial U(s, x; \tau, t)}{\partial n(x)} - U(s, x; \tau, t) t(x, t)] dS(x) dt \\ &= \int_V [U(s, x; \tau, 0) \dot{u}(x, 0) - u(x, 0) \dot{U}(s, x; \tau, 0)] dV(x) \\ &\quad - c^2 \int_0^{\tau^+} \int_S [u(x, t) \frac{\partial U(s, x; \tau, t)}{\partial n(x)} - U(s, x; \tau, t) t(x, t)] dS(x) dt \end{aligned} \quad (7)$$

After using the causality conditions

$$\begin{aligned} U(s, x; \tau, \tau^+) &= 0 \\ \dot{U}(s, x; \tau, \tau^+) &= 0 \end{aligned}$$

$$\begin{aligned} u(x, t) &= \int_0^{\tau^+} \int_V u(s, \tau) (\delta(x-s) \delta(t-\tau)) dV(s) d\tau \\ &= \int_V [u(s, \tau) \dot{U}(x, s; t, \tau) - U(x, s; t, \tau) \dot{u}(s, \tau)] \Big|_{\tau=0}^{\tau^+} dV(s) \\ &\quad - c^2 \int_0^{\tau^+} \int_S [u(s, \tau) \frac{\partial U(x, s; t, \tau)}{\partial n(s)} - U(x, s; t, \tau) t(s, \tau)] dS(s) d\tau \\ &= \int_V [U(x, s; t, 0) \dot{u}(s, 0) - u(s, 0) \dot{U}(x, s; t, 0)] dV(s) \\ &\quad - c^2 \int_0^{\tau^+} \int_S [u(s, \tau) \frac{\partial U(x, s; t, \tau)}{\partial n(s)} - U(x, s; t, \tau) t(s, \tau)] dS(s) d\tau \end{aligned} \quad (8)$$

Since $U(s, x; t, \tau) = U(x, s; t, \tau)$, we have

$$\begin{aligned} u(x, t) &= \int_0^{\tau^+} \int_V u(s, \tau) (\delta(x-s) \delta(t-\tau)) dV(s) d\tau \\ &= \int_V [u(s, \tau) \dot{U}(s, x; t, \tau) - U(s, x; t, \tau) \dot{u}(s, \tau)] \Big|_{\tau=0}^{\tau^+} dV(s) \\ &\quad - c^2 \int_0^{\tau^+} \int_S [u(s, \tau) \frac{\partial U(s, x; t, \tau)}{\partial n(s)} - U(s, x; t, \tau) t(s, \tau)] dS(s) d\tau \\ &= \int_V [U(s, x; t, 0) \dot{u}(s, 0) - u(s, 0) \dot{U}(s, x; t, 0)] dV(s) \\ &\quad - c^2 \int_0^{\tau^+} \int_S [u(s, \tau) \frac{\partial U(s, x; t, \tau)}{\partial n(s)} - U(s, x; t, \tau) t(s, \tau)] dS(s) d\tau \end{aligned} \quad (9)$$

Derivation of D'Alembert' solution by U, T, L, M kernels.

$$u(x, t) = \frac{1}{2} \phi(x+ct) + \frac{1}{2} \phi(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds$$

where

$$u(x, 0) = \phi(x)$$

$$\dot{u}(x, 0) = \psi(x)$$

$$\begin{aligned}U(s, x; t, \tau) &= U(x, s; t, \tau) \\T(s, x; t, \tau) &= \frac{\partial U(s, x; t, \tau)}{\partial n(s)} \\L(s, x; t, \tau) &= \frac{\partial U(s, x; t, \tau)}{\partial n(x)} \\M(s, x; t, \tau) &= \frac{\partial^2 U(s, x; t, \tau)}{\partial n(x) \partial n(s)}\end{aligned}$$

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