EX：Solve the PDE $u_{tt} = c^2 u_{xx}$ ， 0<x<L ， t>0

with initial conditions $u(x,0) = 0$ ， $\dot{u}(x,0) = 0$

and boundary conditions $u(0,t) = \sin(t)$ ， $u(l,t) = \sin(t)$

where c=2 and l=5

(1)Find u(2.5,2.5) by using the diamond rule.

Sol： $\because u(2.5,2.5)$**落於第4區**

$$\therefore u_4(x,t)$$

$$= \sin\left(\frac{x+ct-l}{c}\right) + \sin\left(\frac{ct-x}{c}\right) = \sin\left(\frac{2.5+2\times2.5-5}{2}\right) + \sin\left(\frac{2\times2.5}{2}\right) = 1.8980$$

(2) Find u(2.5,2.5) by using the eigenfunction expansion method.

Sol：

$$u(x,t) = \sum_{n=1}^{3}\left[\frac{-2cl(1-(-1)^n)}{(c^2n^2\pi^2 - l^2)}\sin\left(\frac{n\pi ct}{l}\right) + \frac{2n\pi c^2(1-(-1)^n)}{(c^2n^2\pi^2 - l^2)}\sin(t)\right]\sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore u(2.5,2.5)$$

$$= \sum_{n=1}^{3}\left[\frac{-2\times2\times5(1-(-1)^n)}{(2^2\times n^2\times\pi^2 - 5^2)}\sin\left(\frac{2\times\pi\times c\times t}{l}\right) + \frac{2\times n\times\pi\times2^2(1-(-1)^n)}{(2^2\times n^2\times\pi^2 - 5^2)}\sin(2.5)\right]\sin\left(\frac{n\times\pi\times2.5}{5}\right)$$

$$= 1.8045$$

(3) Find u(2.5,2.5) by using the first order of Cesaro sum.

Sol：

$$u(x,t) = (C,1)\sum_{n=1}^{3}\left[\frac{-2cl(1-(-1)^n)}{(c^2n^2\pi^2 - l^2)}\sin\left(\frac{n\pi ct}{l}\right) + \frac{2n\pi c^2(1-(-1)^n)}{(c^2n^2\pi^2 - l^2)}\sin(t)\right]\sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore u(2.5,2.5)$$

$$= (C,1)\sum_{n=1}^{3}\left[\frac{-2\times2\times5\times(1-(-1)^n)}{(2^2\times n^2\times\pi^2 - 5^2)}\sin\left(\frac{n\times\pi\times2\times2.5}{5}\right) + \frac{2\times n\times\pi\times2^2(1-(-1)^n)}{(2^2\times n^2\times\pi^2 - 5^2)}\sin(2.5)\right]\sin\left(\frac{n\times\pi\times2.5}{5}\right)$$

$$= \frac{3}{3}\left[\frac{-2\times2\times5\times(1-(-1)^1)}{(2^2\times1^2\times\pi^2 - 5^2)}\sin\left(\frac{1\times\pi\times2\times2.5}{5}\right) + \frac{2\times1\times\pi\times2^2(1-(-1)^1)}{(2^2\times1^2\times\pi^2 - 5^2)}\sin(2.5)\right]\sin\left(\frac{1\times\pi\times2.5}{5}\right)]$$

$$+ \frac{2}{3}\left[\frac{-2\times2\times5\times(1-(-1)^2)}{(2^2\times2^2\times\pi^2 - 5^2)}\sin\left(\frac{2\times\pi\times2\times2.5}{5}\right) + \frac{2\times2\times\pi\times2^2(1-(-1)^2)}{(2^2\times2^2\times\pi^2 - 5^2)}\sin(2.5)\right]\sin\left(\frac{2\times\pi\times2.5}{5}\right)]$$

$$+ \frac{1}{3}\left[\frac{-2\times2\times5\times(1-(-1)^3)}{(2^2\times3^2\times\pi^2 - 5^2)}\sin\left(\frac{3\times\pi\times2\times2.5}{5}\right) + \frac{2\times3\times\pi\times2^2(1-(-1)^3)}{(2^2\times3^2\times\pi^2 - 5^2)}\sin(2.5)\right]\sin\left(\frac{3\times\pi\times2.5}{5}\right)]$$

$$= 1.9867$$

(4) Find u(2.5,2.5) by using the quasi-static decomposition method.

Sol：

$$u(x,t)=\sin(t)+\sum_{n=1}^{3}\frac{-2l}{(c^2n^2\pi^2-l^2)}(1-(-1)^n)\times(c\sin(\frac{n\pi ct}{l})-\frac{l}{n\pi}\sin(t))\sin(\frac{n\pi x}{l})$$

$$\therefore u(2.5,2.5)=$$

$$\sin(2.5)+\sum_{n=1}^{3}\frac{-2\times 5}{(2^2\times n^2\times\pi^2-5^2)}(1-(-1)^n)\times(2\times\sin(\frac{n\times\pi\times 2\times 2.5}{5})-\frac{l}{n\times\pi}\sin(2.5))\sin(\frac{n\times\pi\times 2.5}{5})$$

$$=1.895$$

(5) Find u(2.5,2.5) by using the Stokes' transformation.

Sol：

$$u(x,t)=\sin(t)+\sum_{n=1}^{3}[\frac{-2cl}{(c^2n^2\pi^2-l^2)}\sin(\frac{n\pi ct}{l})+\frac{2n\pi c^2}{(c^2n^2\pi^2-l^2)}\sin(t)-\frac{-2\sin(t)}{n\pi}](1-(-1)^n)\sin(\frac{n\pi x}{l})$$

$$\therefore u(2.5,2.5)=\sin(2.5)$$

$$+\sum_{n=1}^{3}[\frac{-2\times 2\times 5}{(2^2\times n^2\times\pi^2-5^2)}\sin(\frac{n\times\pi\times 2\times 2.5}{5})+\frac{2\times n\times\pi\times 2^2}{(2^2\times n^2\times\pi^2-5^2)}\sin(2.5)-\frac{-2\times\sin(2.5)}{n\times\pi}](1-(-1)^n)\sin(\frac{n\times\pi\times 2.5}{5}$$

$$=1.895$$

Table 1：Solutions for a string subjected to multi-support motions

Exact solution	Diamond rule	Eigenfunction expansion method	Cesaro sum	Quasi-static decomposition method	Stokes' transformation
	1.8980	1.8045	1.9867	1.895	1.895
Error (0%)	0%	4.93%	4.67%	0.16%	0.16%