Engineering Mathematics (A) — M.S. Program Ent. Exam. 2004

1. (17%) Let

$$\mathbf{A} = \begin{bmatrix} 89 & 48 & 0 \\ b & 61 & 0 \\ c & 0 & 1 \end{bmatrix}.$$

   (a) Find the values of b and c such that the eigenvalues of $\mathbf{A}$ are all real and the eigenvectors are orthogonal to each other.

   (b) For such b and c find the eigenvalues of $\mathbf{A}$ and the corresponding orthogonal diagonalizing matrix $\mathbf{P}$ and also its inverse matrix $\mathbf{P}^{-1}$.

2. (16%) Let real-valued functions $g(x)$ and $h(x)$ defined over $0 \le x \le 1$ be expanded as

$$g(x) = \sum_{n=-\infty}^{\infty} G_n e_n(x), \quad h(x) = \sum_{n=-\infty}^{\infty} H_n e_n(x),$$

where $e_0(x) := 1, \quad e_n(x) := \sqrt{2}\cos(2n\pi x), \quad e_{-n}(x) := \sqrt{2}\sin(2n\pi x)$ for $n = 1, 2, 3, ...$, and the coefficients G_n and H_n $(n = ..., -2, -1, 0, 1, 2, ...)$ are real numbers.

   (a) Show that $\{..., e_{-2}(x), e_{-1}(x), e_0(x), e_1(x), e_2(x), ...\}$ are a set of orthonormal functions defined over $0 \le x \le 1$.

   (b) Find a formula relating $\int_0^1 g(x)h(x)dx$ to $\sum_{-\infty}^{\infty} G_n H_n$ and show clearly how to derive the formula.

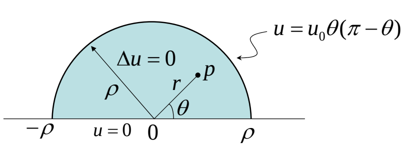
3. (18%) Solve the following initial value problems:

   (a) (8%) $\frac{dy}{dx} - 2y^2 + 3y = 1, \quad y(0) = 1;$

   (b) (6%) $\frac{dy}{dx} = \frac{x+4y}{x}, \quad y(0) = 0;$

   (c) (4%) $\frac{dy}{dx} = \frac{x+4y}{x}, \quad y(0) = 1.$

4. (15%) Find the steady state temperature distribution in the semicircular region of radius ρ lying in the upper half-plane and centered on the origin, as shown in the figure. The temperature on the straight boundary is $u = 0$, and that on the semicircular boundary is $u = u_0\theta(\pi - \theta)$.



5. (17%) Complex analysis.

   (a) (7%) State the Cauchy Integral Formula and the Laurent Expansion Theorem.

   (b) (10%) Find all the possible Laurent series about $z = 0$ for the complex-valued function $f(z) = \frac{1}{1-z}$, $(z = x + iy)$, by using the Laurent Expansion Theorem only.

6. (17%) Calculus of variations.

   (a) (10%) Find the extremal for the functional $v(y(x)) = \int_0^1 \frac{\sqrt{1+(\frac{dy}{dx})^2}}{x} dx$ with boundary conditions $y(0) = 1$ and $y(1) = 0$.

   (b) (7%) Find the transversality condition for the functional $v(y(x)) = \int_0^{x_1} \frac{\sqrt{1+(\frac{dy}{dx})^2}}{x} dx$ with boundary conditions $y(0) = 1$ and (x_1, y_1) constrained on a given curve $y_1 = \phi(x_1)$.