法一：Trefftz method

由 Trefftz method 我們可將場解表示式表示為

$$u(r,\theta)=a+b\ln r+\sum_{n=1}^{\infty}(c_n r^n\cos n\theta+d_n r^n\sin n\theta+e_n\frac{1}{r^n}\cos n\theta+f_n\frac{1}{r^n}\sin n\theta)$$

由於此為內域問題且當 $r:-\rho\to 0\to\rho$ 時 $u=0$

所以 $u(r,\theta)$ 可被簡化成

$$u(r,\theta)=\sum_{n=1}^{\infty}d_n r^n\sin n\theta$$

帶入邊界條件後得

$$u(\rho,\theta)=\sum_{n=1}^{\infty}d_n\rho^n\sin n\theta=u_0\theta(\pi-\theta)$$

令 $u_0=1$，再由正交函數

$$d_n\rho^n\int_0^{\pi}\sin n\theta\sin n\theta d\theta=\int_0^{\pi}\theta(\pi-\theta)\sin n\theta d\theta$$

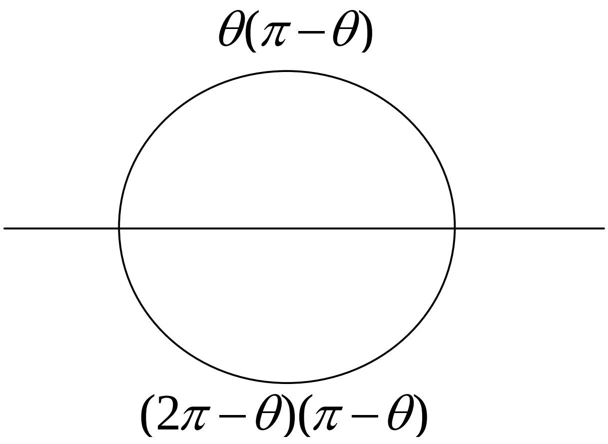
$$d_n=\frac{1}{\rho^n}\frac{2}{\pi}\frac{2-2\cos n\pi}{n^3}=\frac{4+4(-1)^n}{\rho^n n^3\pi}$$

令 $\rho=1$,可得場解表示式為

$$u(r,\theta)=\sum_{n=1}^{\infty}\frac{4-4(-1)^n}{n^3\pi}r^n\sin n\theta$$

法二：Boundary Integral Equation Method

將原本半圓問題，轉換成圓之問題



Laplace problem 之退化核為

$$U(s,x)=\begin{cases} U^{I}(R,\theta;\rho,\phi)=\ln R-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R})^{m}\cos(m(\theta-\phi)), & R>\rho \\ U^{E}(R,\theta;\rho,\phi)=\ln \rho-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{R}{\rho})^{m}\cos(m(\theta-\phi)), & R<\rho \end{cases}$$

$$T(s,x)=\begin{cases} T^{I}(R,\theta;\rho,\phi)=\frac{1}{R}+\sum_{m=1}^{\infty}\frac{1}{R}(\frac{\rho}{R})^{m}\cos(m(\theta-\phi)), & R>\rho \\ T^{E}(R,\theta;\rho,\phi)=-\sum_{m=1}^{\infty}\frac{1}{R}(\frac{R}{\rho})^{m}\cos(m(\theta-\phi)), & R<\rho \end{cases}$$

由 null-field integral equation: (令圓之半徑為 1)

$$0=\int(T^{E}u-U^{E}t)dB(s)$$

$$\Rightarrow\int T^{E}u\ dB(s)=\int U^{E}t\ dB(s)$$

$$\int T^{E}u\ dB(s)=\int_{0}^{\pi}T^{E}u\ dB(s)+\int_{\pi}^{2\pi}T^{E}u\ dB(s)$$

$$=-\sum_{m=1}^{\infty}\frac{4}{R}(\frac{R}{\rho})^{m}\frac{[m\pi\cos(\frac{m\pi}{2})-2\sin(\frac{m\pi}{2})]\sin(\frac{m\pi}{2})[\sin(m\pi)\cos(m\phi)-\sin(m\phi)\cos(m\pi)]}{m^{3}}$$

$$=-\sum_{m=1}^{\infty}\frac{4}{R}(\frac{R}{\rho})^{m}\frac{[m\pi\cos(\frac{m\pi}{2})\sin(\frac{m\pi}{2})-2\sin^{2}(\frac{m\pi}{2})][-(-1)^{m}\sin(m\phi)]}{m^{3}}$$

$$\int_0^{2\pi} U^E t\, dB(s) = \int_0^{2\pi} (\ln\rho - \sum_{m=1}^{\infty} \frac{1}{m} (\frac{R}{\rho})^m \cos(m(\theta-\phi)))(a_0 + \sum_{n=1}^{\infty} a_n \cos(m\theta) + b_n \sin(n\theta)) dB(s)$$

$$= 2\pi\, a_0 \ln\rho - \sum_{m=1}^{\infty} a_m \frac{\pi}{m} (\frac{R}{\rho})^m \cos(m\phi) + b_m \frac{\pi}{m} (\frac{R}{\rho})^m \sin(m\phi)$$

比較係數可得:

$a_0 = 0$

$a_n = 0$

$$b_n = \frac{4-4\cos m\pi}{R\, m^2 \pi}$$

$$t = \frac{4-4\cos m\pi}{R\, m^2 \pi} \sin(m\theta))$$

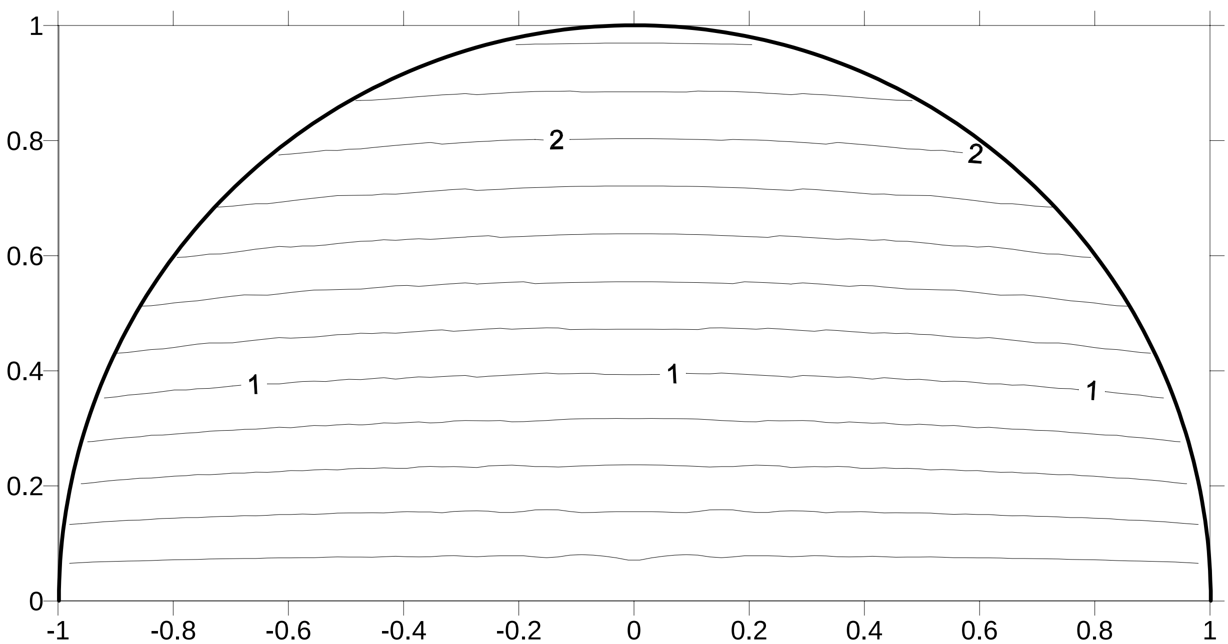
由域內點積分方程可得

$$2\pi u(x) = \int_B [T^I(s,x)u(s) - U^I(s,x)t(s)]dB(s)$$

$$= \sum_{m=1}^{\infty} (\frac{\rho^m}{R^{m+1}})[\frac{8-8\cos(m\pi)}{m^3} \sin(m\phi)]$$

$$\Rightarrow u(x) = \frac{1}{\pi} \sum_{m=1}^{\infty} (\frac{\rho^m}{R^{m+1}})[\frac{4-4(-1)^m}{m^3} \sin(m\phi)]$$

因為 source point 佈於邊界上，所以 $R=1$，所得之場解表示式即予用 Trefftz 所得之解相同。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1
1