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1. Classify the (a). ordinary differential equation , (b). integral equation, (c). integro-differential equation, (d). logic equation and (e). linear algebraic equation. (20 %)

Equation	Equation type (a, b, c, d, e)
$y''(t) + 4y(t) = 0$	a
$x^2 + 3x + 9 = 0$	e
$y(t) = \int_0^t y(s)ds$	b
$y'(t) = \int_0^t y(s)ds$	c
$A \bigcup B = C$	d

(註: 請將本表填入 a, b, c, d, e 後, 抄入答案卷才計分)

2. Please explain the Green's theorem (5 %) and the Green's function. (5 %)
Sol:
Green's theorem:

$$\oint Pdx + \oint Qdy = \int\int (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$$

Green's function: $G(x, s)$表示在s施一集中力, 而在x產生之反應.

3. Given an anti-symmetric matrix W as follows:

$$W = \begin{bmatrix} 0 & \frac{-2}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{-2}{3} \\ \frac{-1}{3} & \frac{2}{3} & 0 \end{bmatrix}.$$

(a). Calculate $W^T + W =$? (3 %)
Sol: 零矩陣
(b). Calculate $W^3 + W =$? (3 %)
Sol: 零矩陣
(c). Calculate the determinant for W. (4 %)
Sol: 0
where the superscript T denotes transpose.

4. Based on the following relations of the Laplace transform ($\mathcal{L}$),

$$\mathcal{L}\{ty(t)\} = -Y'(s), \ \ where \ \ \mathcal{L}\{y(t)\} = Y(s)$$

the following second order ODE

$$t^2\ddot{y}(t) - 4t\dot{y}(t) + 6y(t) = 0$$

can be transformed to (transform $y(t)$ to $Y(s)$):

$$s^2Y''(s) + asY'(s) + bY(s) = 0$$

where $Y(s)$ is the Laplace transform of $y(t)$, determine a and b. (5 %)
Sol: $a = 8, b = 12$

If we repeat the Laplace transform with respect to $Y(s)$ again (transform $Y(s)$ to $\bar{Y}(v)$) where $\bar{Y}(v)$ must satisfy

$$v^2\frac{d^2\bar{Y}(v)}{dv^2} + pv\frac{d\bar{Y}(v)}{dv} + q\bar{Y}(v) = 0$$

determine p and q. (5 %)
Sol: $p = -4, q = 6$

5. Stokes's theorem (transformation between surface integrals and line integrals)

Let S be a piecewise smooth oriented surface in space and let the boundary S be a piecewise smooth simple closed curve C. Let $\mathbf{F}(x, y, z)$ be a continuous vector function that has continuous first partial derivatives in a domain in space containing S. Then,

$$\int_S (curl\,\mathbf{F})\cdot \mathbf{n} dA = \oint_C \mathbf{F}\cdot\mathbf{r}' dS$$

where $\mathbf{n}$ is a unit normal vector of S and dependng on $\mathbf{n}$, the integration around C is taken in the sense shown in Fig.1, also $\mathbf{r}' = d\mathbf{r}/ds$ in the unit tangent normal vector and s is the arc length of C.

(a). Write down the physical interpretation of the Stokes's theorem. (3 %)
Sol:

$$\int_S\int \nabla\times F\cdot n\ dA = \oint_s F\cdot r'(s)ds$$

(b) Verify the Stokes's theorem for $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ and S the paraboloid $z = f(x, y) = 1-(x^2+y^2), z \geq 0$ in Fig.2. (10 %)

Fig.1

Fig.2

Sol:

$$r(s) = cos(s)i + sin(s)j, r'(s) = -sin(s)i + cos(s)j$$

$$\oint F\cdot dr = \int_0^{2\pi}[(sin(s))(-sin(s)) + 0 + 0]ds = -\pi......\text{Ans.}$$

$$\nabla\times F = curl\ \ F = -i - j - k$$

$$N = f_x\ \ i - f_y\ \ j + k = 2x\ \ i + 2y\ \ j + k$$

$$\nabla\times F\cdot N = -2x - 2y - 1$$

$$\int_S\int \nabla\times F\cdot ndA = \int_0^{2\pi}\int_0^1(-2rcos(\theta) - 2rsin(\theta) - 1)rdrd\theta = -\pi......\text{Ans.}$$

6. By taking the Fourier transform of the equation $\frac{d^2\phi}{dx^2} - K^2\phi = f(x)$, show that its solution $\phi(x)$ can be written as

$$\phi(x) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ikx}\bar{f}(k)}{k^2 + K^2} dk,$$

where $\bar{f}(k)$ is the Fourier transform of $f(x)$. (10 %)

$$\mathcal{F}\{\frac{d^2\phi}{dx^2} - K^2\phi = f(x)\}$$

$$(-ik)^2\bar{\phi}(k)K^2\bar{\phi}(k) = -k^2\bar{\phi}(k) - K^2\bar{\phi}(k) = \bar{f}(k)$$

$$\bar{\phi}(k) = \frac{-1}{k^2 + K^2}\bar{f}(k)$$

$$\phi(k) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ikx}\bar{f}(k)}{k^2 + K^2} dk$$

7. Given the one-dimensional heat equation with initial and boundary conditions,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x)$$

$$u_x(0,t) = 0, \text{ for all t}$$

$$u_x(L,t) = 0, \text{ for all t}$$

where c^2 is the thermal diffusivity, L is the bar length, x is space, t is time and $u(x,t)$ is temperature.
(a). Write out the physical meaning of the one-dimensional heat equation. (3 %)
Sol: the principle of conservation of energy law.
(b). Find a solution of the one-dimensional heat equation using the method of separating variables (or product method). (10 %)

$$\because u(x,t) = F(x)G(t)$$

$$u_x(0,t) = F'(0)G(t) = 0, u_L(L,t) = F'(L)G(t) = 0$$

$$F(x) = A\ \ cos(px) + B\ \ sin(px) \Rightarrow F'(x) = -Ap\ \ sin(px) + B\ \ pcos(px)$$

$$F'(0) = Bp = 0, F'(L) = -Ap\ \ sin(pL) = 0 \Rightarrow p = p_n = \frac{n\pi}{L}, n = 0, 1, 2, 3, ...$$

$$A = 1, B = 0 \Rightarrow F_n(x) = cos(\frac{n\pi x}{L}), n = 0, 1, 2, 3, ...$$

$$u_n(x,t) = F_n(x)G_n(t) = A_n cos(\frac{n\pi x}{L})e^{-\lambda_n^2 t}, \lambda_n = \frac{c_n\pi}{L}, \lambda_0 = 0 \Rightarrow u_0 = constant$$

Fourier cosine series

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(x,t) = \sum_{n=-\infty}^{\infty} A_n cos(\frac{n\pi x}{L})e^{-\lambda_n^2 t}$$

$$u(x,0) = \sum_{n=-\infty}^{\infty} A_n cos(\frac{n\pi x}{L}) = f(x)$$

where

$$A_0 = \frac{1}{L}\int_0^L f(x)dx$$

$$A_n = \frac{2}{L}\int_0^L f(x)cos(\frac{n\pi x}{L})dx, n = 1,2,3...$$

8. Using

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum Res \ \ f(z),$$

show that (14 %)

$$\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}.$$

Sol:

$$f(z) = \frac{1}{1+x^4} \Rightarrow z_1 = e^{\frac{\pi i}{4}}, z_2 = e^{\frac{3\pi i}{4}}, z_3 = e^{-\frac{3\pi i}{4}}, z_4 = e^{-\frac{\pi i}{4}} \text{(four poles)}$$

$$Res_{z=z_1} f(z) = [\frac{1}{(1+z^4}))]_{z=z_1} = (\frac{1}{4z^3})_{z=z_1} = \frac{1}{4}e^{-\frac{3\pi i}{4}} = -\frac{1}{4}e^{\frac{\pi i}{4}}$$

$$Res_{z=z_2} f(z) = [\frac{1}{(1+z^4)}]_{z=z_2} = (\frac{1}{4z^3})_{z=z_1} = \frac{1}{4}e^{-\frac{9\pi i}{4}} = -\frac{1}{4}e^{-\frac{\pi i}{4}}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+z^4}dx = \frac{2\pi i}{4}(-e^{\frac{\pi i}{4}} + e^{-\frac{\pi i}{4}}) = \pi sin(\frac{\pi}{4}) = \frac{\pi}{\sqrt{2}}$$

$\because f(x) = \dfrac{1}{1+x^4}$ is even function,

$$\therefore \int_{\infty}^{\infty} \frac{1}{1+x^4}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^4}dx = \frac{\pi}{2\sqrt{2}}$$