國立臺灣海洋大學河海工程研究所 1996 工程數學入學考

1. Find the following improper integrals in principal values (15 %)

$$\int_{-1}^{1} ln \mid x \mid dx =?, \int_{-1}^{1} \frac{1}{x} dx =?, \int_{-1}^{1} \frac{1}{x^2} dx =?.$$

2. Given

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

(1). Find eigenvalues of A. (5 %)
(2). Find A^{100}. (5 %)
(3). Find eigenvalues of A^{100}. (5 %)

3. Solve the PDE of an infinite string $-\infty < x < \infty$. (10%)

$$u_{tt} = \begin{cases} 4u_{xx}, & for \;\; x < 0, \;\; t > 0 \\ 1u_{xx}, & for \;\; x > 0, \;\; t > 0, \end{cases}$$

and initial conditions

$$u(x, 0) = 0, u_t(x, 0) = 0,$$

and $u(x, t)$ is continuous across $x = 0$, while

$$u_x(0^+, t) - u_x(0^-, t) = a \; sin(\omega t),$$

where a and ω are two constants.

4. If the real and imaginary parts of $\bar{X}(\bar{\omega})$ are defined as follows:(10 %)

$$\bar{X}(\bar{\omega}) = \frac{1}{(-\bar{\omega}^2 + 2i\xi\omega\bar{\omega} + \omega^2)} = \bar{X}_R(\bar{\omega}) + \bar{X}_I(\bar{\omega})i.$$

That is

$$\bar{X}_R(\bar{\omega}) = \frac{\omega^2 - \bar{\omega}^2}{(\omega^2 - \bar{\omega}^2)^2 + 4\xi^2\omega^2\bar{\omega}^2}, \;\; \bar{X}_I(\bar{\omega}) = \frac{-2\xi\omega\bar{\omega}}{(\omega^2 - \bar{\omega}^2)^2 + 4\xi^2\omega^2\bar{\omega}^2}$$

where ω and ξ are constants and $0 < \xi < 1$. Prove the following indentities by using the theory of residue.

$$-\bar{X}_I(\bar{\omega}) = \int_{-\infty}^{\infty} \frac{\bar{X}_R(u)}{\pi(\bar{\omega} - u)} du, \;\; \bar{X}_R(\bar{\omega}) = \int_{-\infty}^{\infty} \frac{\bar{X}_I(u)}{\pi(\bar{\omega} - u)} du.$$

(Hint: Prove both cases together by multiplying $i = \sqrt{-1}$ for the second case and treat u as a complex variable.)

5. Find a general solution of (13 %)

$$y'' - \frac{4}{x} y' + \frac{4}{x^2} y = x^2 + 1.$$

6. Given $y'' - xy' + y = -xcos(x)$, find two independent power series about the ordinary point $x = 0$ that are solution to the given differential equation. (13 %)

7. Solve the differential equation (12 %)

$$y'' + 2y' + y = \delta(t-1); y(0) = 2, y'(0) = 3$$

where δ is the Dirac Delta function.

8. Prove that $J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ixsin(\phi)} d\phi$ in which J_0 is the Bessel function of the first kind of order 0. (12 %)