1.(25%) $\mathbf{x} = (x_1, x_2, x_3)^T$ and $\mathbf{y} = (y_1, y_2, y_3)^T$ are three-dimensional column vectors. The superscript T denotes the transpose.
(a)(5%) Write the 3×3 matrix $\mathbf{x}\mathbf{y}^T$.
(b)(10%) Let $\mathbf{A} = \mathbf{I}_3 + \mathbf{x}\mathbf{y}^T$, where $\mathbf{I}_3$ is a third order unit matrix. Under what condition that $\mathbf{A}$ is invertible.
(c)(10%) If the above condition holds, derive $\mathbf{A}^{-1}$ in terms of $\mathbf{x}\mathbf{y}^T$ and $\mathbf{y}^T\mathbf{x}$, the latter of which is the inner product of $\mathbf{x}$ and $\mathbf{y}$.

2.(30%) (a)(10%) $y_1 = 1$ is a particular solution of the following Riccati differential equation:

$$y' = \frac{1}{x}y^2 + \frac{1}{x}y - \frac{2}{x}, \quad x > 0. \tag{1}$$

Let $y(x) = y_1 + z(x)$ to derive the general solution of Eq. (1).
(b)(10%) Under the variable transformation

$$y = \frac{-xu'}{u}, \quad x > 0 \tag{2}$$

can you derive the corresponding second order differential equation for $u(x)$. What is the particular solution $u_1(x)$ of the equation you derived, and please use the reduction of order method to find another particular solution $u_2(x)$. Do they pass the Wronskian test?
(c)(10%) Employ the inverse operator method to solve

$$(D^2 + 2D + 1)y = x \sin x.$$

3.(45%) $\boldsymbol{\nabla}$ is the del operator. $\mathbf{A}$, $\mathbf{B}$ are vector fields and ϕ is a scalar field. The cross product of $\mathbf{A}$ and $\mathbf{B}$ is written as $(\mathbf{A} \times \mathbf{B})_i = \varepsilon_{ijk}A_jB_k$, and the curl of $\mathbf{A}$ is written as $(\boldsymbol{\nabla} \times \mathbf{A})_i = \varepsilon_{ijk}A_{k,j}$, where ε_{ijk} is the permutation symbol, and $A_{k,j} = \partial A_k / \partial x_j$. Note the $\varepsilon - \delta$ identity: $\varepsilon_{ijk}\varepsilon_{kmn} = \varepsilon_{kij}\varepsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$. Derive
(a)(5%) $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = \mathbf{0}$.
(b)(5%) $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$.
(c)(10%) $\mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}\|\mathbf{A}\|^2/2 - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{A}$, where $\|\mathbf{A}\|$ is the norm of $\mathbf{A}$.
(d)(10%) $\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} - \mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{A}) - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B})$.
(e)(15%) Consider the Navier-Stokes equation for a fluid of constant density and viscosity:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}\left(\frac{p}{\rho}\right) + \nu\boldsymbol{\nabla}^2\mathbf{u}. \tag{3}$$

By using the results in (a)-(d), derive the vorticity equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{u} + \nu\boldsymbol{\nabla}^2\boldsymbol{\omega}, \tag{4}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$.