Part II

Acoustic eigenproblem

Outlines

1. Introduction
2. 2-D acoustic eigenproblem
3. SVD updating terms
4. SVD updating documents
5. 3-D cavity
6. Conclusions

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Eigenproblem

$$u''(x) = \lambda u(x)$$

$$\nabla^2 u(x) = \lambda u(x)$$

$$\nabla^4 u(x) = \lambda u(x)$$

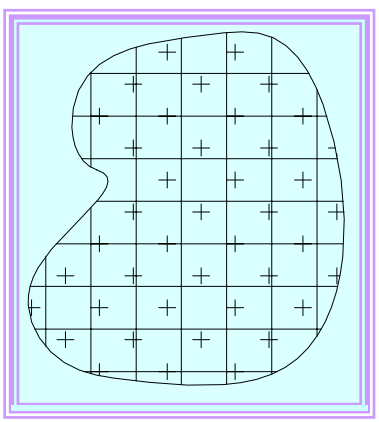
$$\int K(x,s)u(s)ds = \lambda u(x)$$

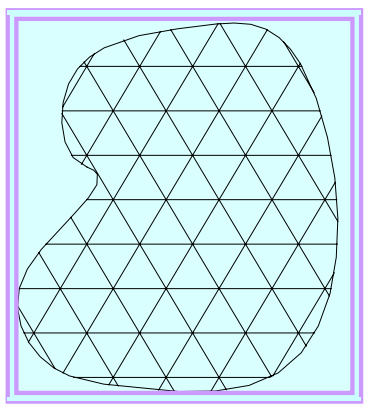
$$[A]\{x\} = \lambda\{x\}$$

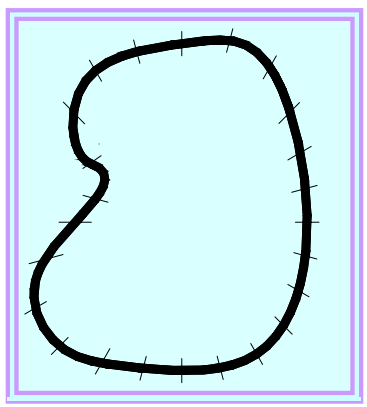
$$[K(\lambda)]\{x\} = 0$$

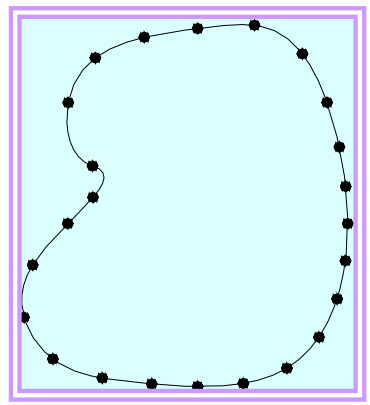
Numerical methods

- Numerical Methods
  - Mesh Methods
    - Finite Difference Method
    - Finite Element Method
    - Boundary Element Method
  - Meshless Method



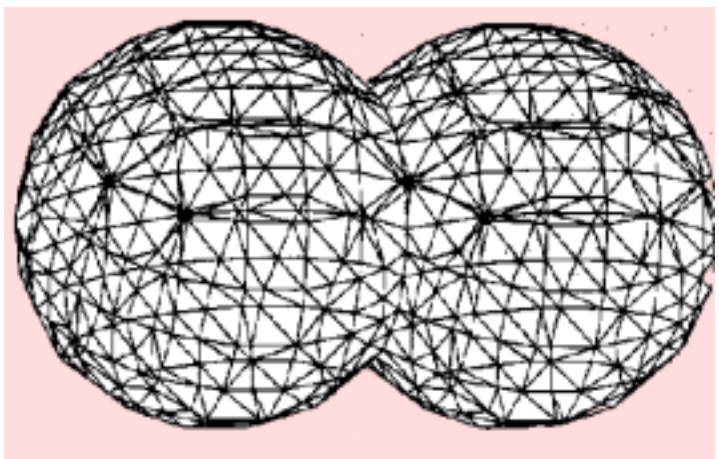




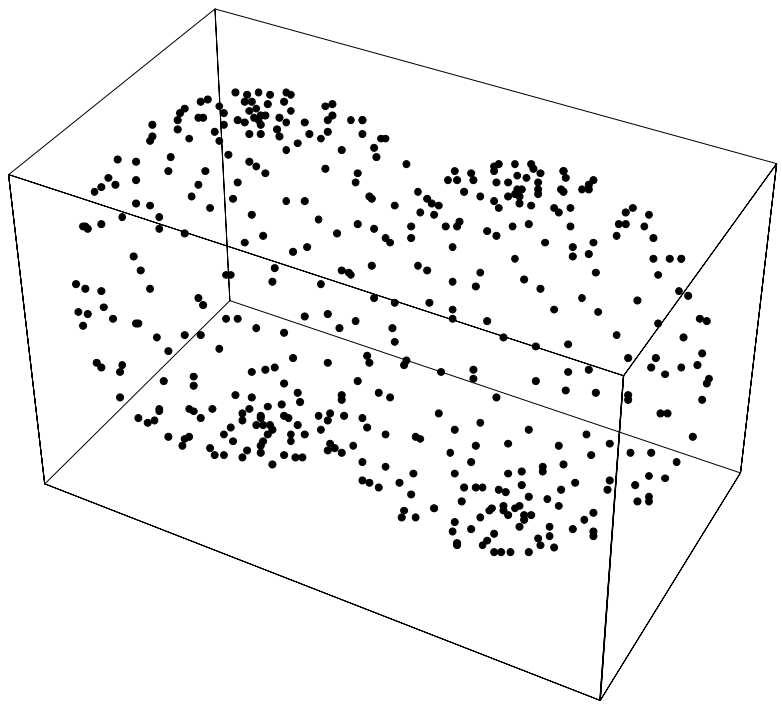


What is meshless method ?

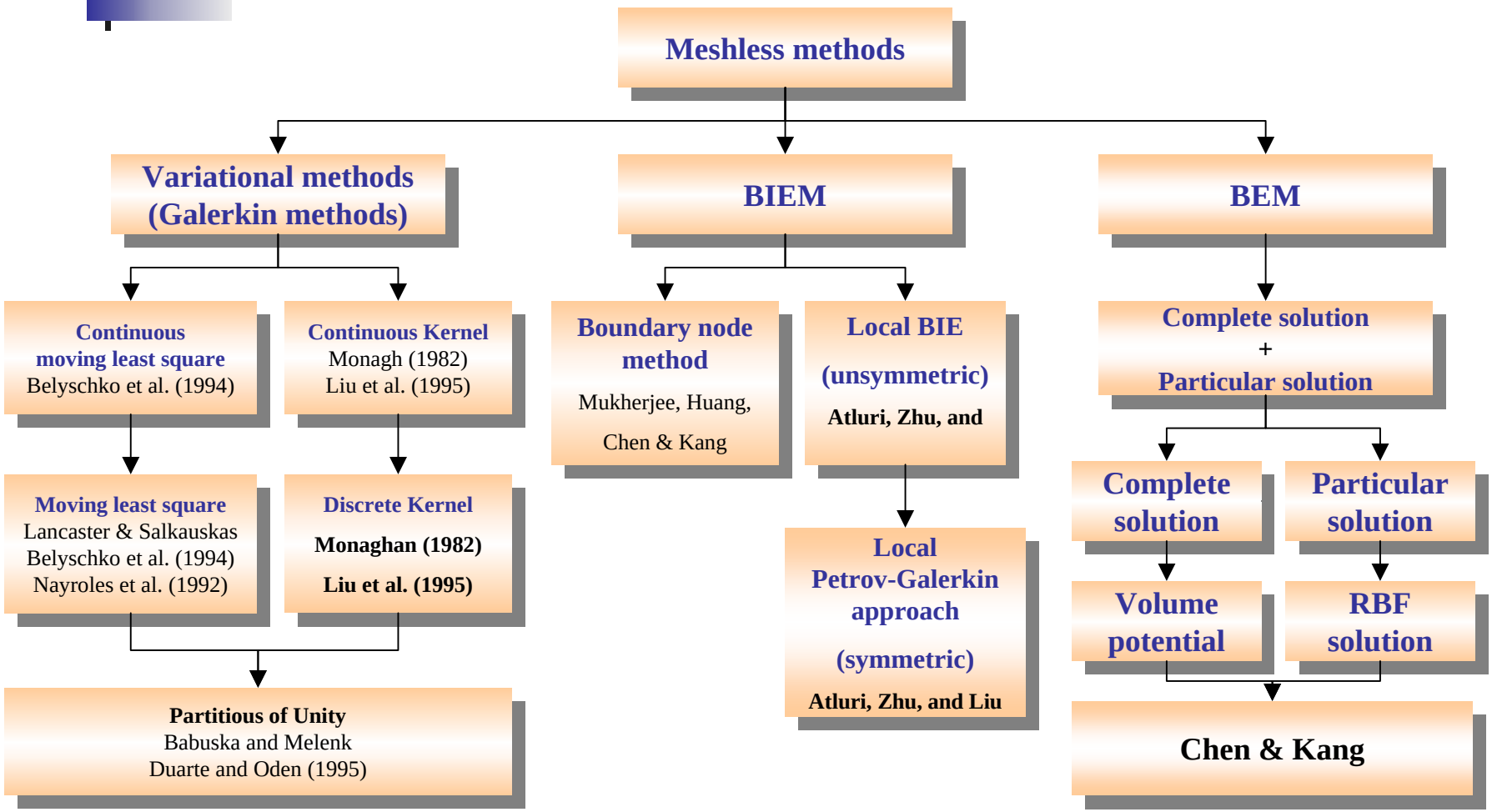
MESH METHOD
(BEM or FEM)



MESHLESS METHOD



The developments of meshless methods



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The developments of radial basis functions

Radial basis function (RBFs)

- **Mesh method**
  - **Globally-supported RBFs**
    - **DRBEM** Nardini Brebbia 1982 → $\psi(r) = 1 + r$
    - **Method of particular integral** Ahmad & Banerjee 1986 → $\psi(r) = C - r$
    - **Equivalence** Polyzos &Beskos 1994
- **Meshless method**
  - **Compactly-supported RBFs** Wu、Buhmann & Wendland
    - $\psi(r) = (1-r)_+^4 (4 + 16r + 12r^2 + 3r^3)$
    - $\psi(r) = (1-r)_+^6 (6 + 36r + 82r^2 + 72r^3 + 30r^4 + 5r^5)$
    - $\psi(r) = \frac{1}{3} + r^2 - \frac{4}{3}r^3 + 2r^2 \ln r$
    - $\psi(r) = \frac{1}{15} + \frac{19}{6}r^2 - \frac{16}{3}r^3 + 3r^4 - \frac{16}{15}r^5 + \frac{1}{6}r^6 + 2r^2 \ln r$
    - $\psi(r) = (1-r)_+^6 (3 + 18r + 35r^2)$
    - $\psi(r) = (1-r)_+^8 (1 + 8r + 25r^2 + 32r^3)$
  - **Globally-supported RBFs**
    - **Volume potential** → **Method of fundamental solution** → $\psi(r) = U_c(s,x) = U_c(|x-s|) = U_c(r)$ **(Potential theory)**
    - **Present method**
      - $1D: \psi(r) = \mathrm{Im}\{U_c(s,x)\} = \frac{-\cos(kr)}{2k}$
      - $2D: \psi(r) = \mathrm{Im}\{U_c(s,x)\} = J_0(kr)$
      - $3D: \psi(r) = \mathrm{Im}\{U_c(s,x)\} = \frac{\sin(kr)}{r}$

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Introduction of Radial Basis Functions

a. What is radial basis function?

$$\varphi(r) = \varphi\|x - x_k\|,$$

where x is the field point and x_k is the interpolation point.

b. Scattered data interpolation (Software: Surfer, Grapher…)

$$g(x) = \sum_{k=1}^{N} c_k \varphi(\|x\text{-}x_k\|) \text{ ; where } \varphi(\|x\text{-}x_k\|) \text{ is radial basis function.}$$

c. This problem could lead to a linear system $\mathrm{Ac} = \mathrm{f}$.

$$A_{jk} = \varphi(\|x_j\text{-}x_k\|), \quad j,k = 1, 2, \cdots, n$$

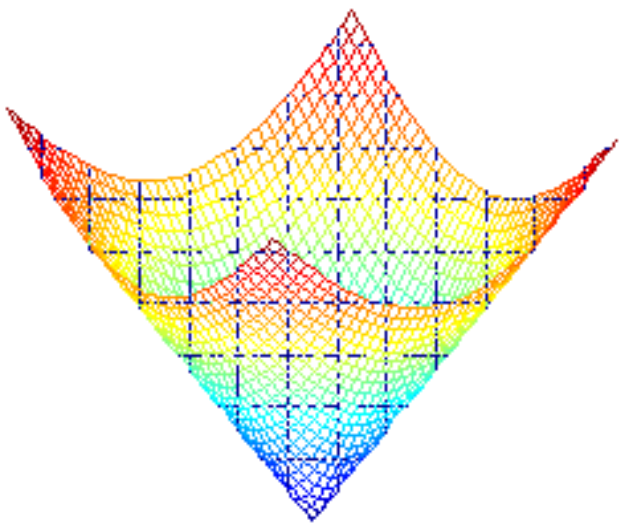
Globally Supported Radial Basis Functions

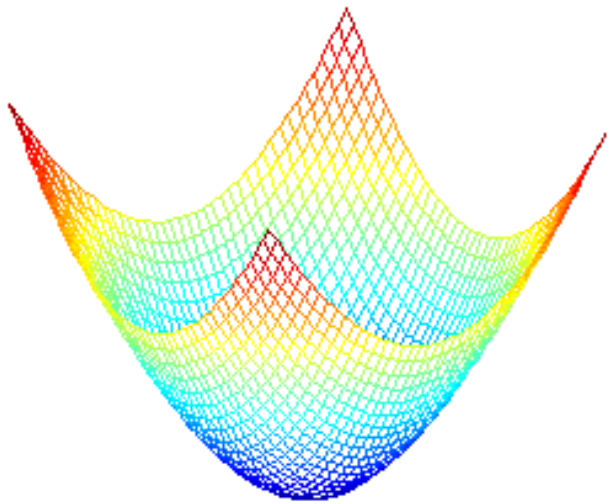
a. Globally Supported Radial Basis Functions (GSRBF)

$$\varphi = 1 + r$$

$$\varphi = \sqrt{r^2 + c^2}$$

(DRBEM or Method of Particular Integrals)

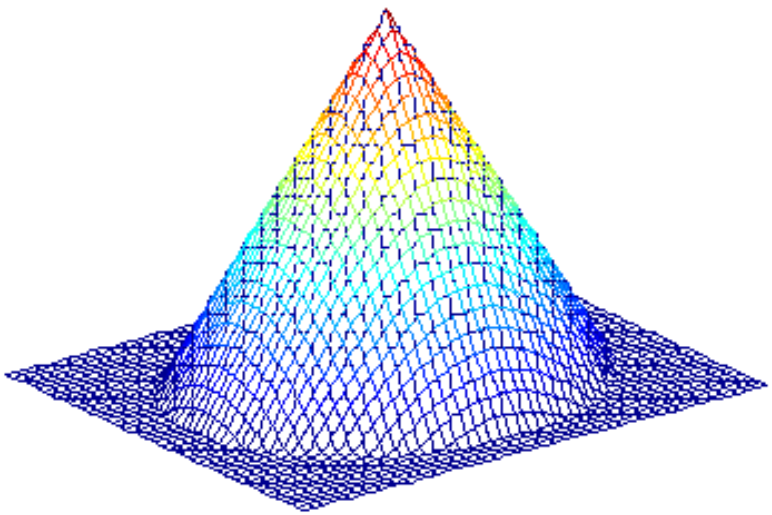




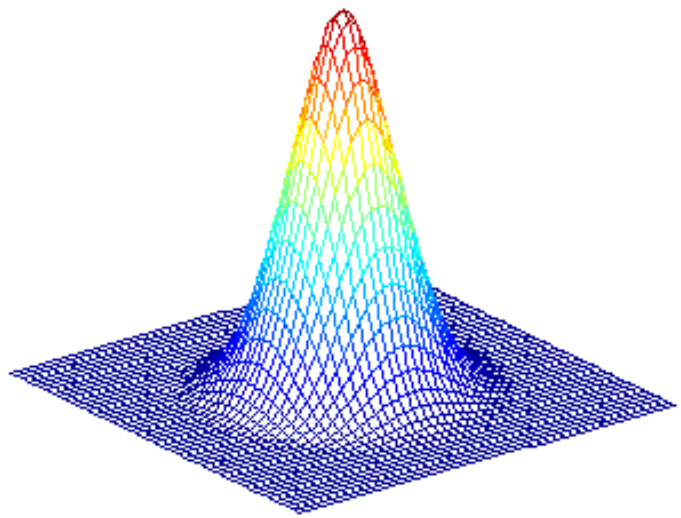
Compactly Supported Radial Basis Functions

b. Compactly Supported Radial Basis Functions (CSRBF)

$$\varphi = (1-r)_+^2$$



$$\varphi = (1-r)_+^4 (4r+1)$$



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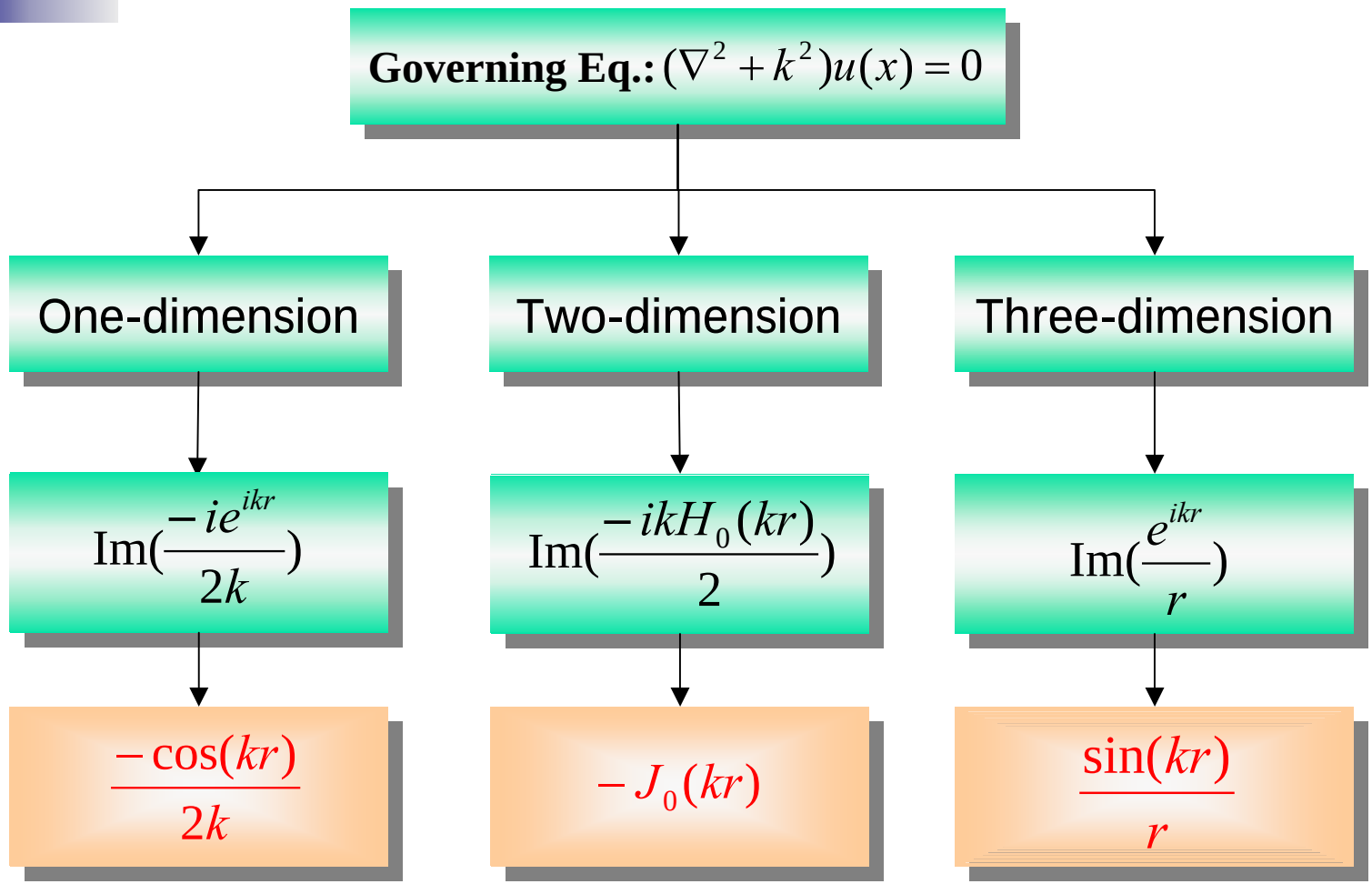
RBF in the MFS

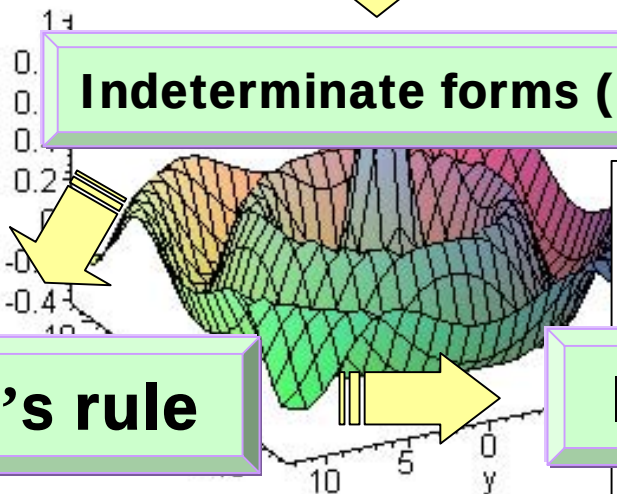


The derivation of indeterminate forms

RBF for $J_0(kr)$

Influence matrices

Indeterminate forms ($\frac{0}{0}$)



L'Hôpital's rule

$$\lim_{s \to x} U(s,x) = 1$$

$$\lim_{s \to x} T(s,x) = 0$$

Invariant method

$$\lim_{s \to x} M(s,x) = \frac{k^2}{2}$$

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The diagonal elements of U kernel

$[U(s,x)]$

$$\lambda_\ell = 2NJ_\ell(k\ell)J_\ell(k\ell), \quad \ell = 0, \pm 1, \cdots, \pm(N-1), N.$$

$$2Na_0 = 2N\sum_{\ell=-\infty}^{\ell=\infty} J_\ell(k\ell)J_\ell(k\ell).$$

$$+$$

$$1 = J_0^2(k\ell) + 2\sum_{m=-\infty}^{\infty} J_m^2(k\ell).$$

Addition theorem of Bessel function

$diag\ ([U(s,x)])$

$$a_0 = 1.$$

The diagonal elements of T kernel

$[T(s,x)]$

$$\mu_\ell = 2NJ'_\ell(k\ell)J_\ell(k\ell), \quad \ell = 0, \pm 1, \cdots, \pm(N-1), N.$$

$$2Nb_0 = 2N\,k\sum_{\ell=-\infty}^{\ell=\infty} J'_\ell(k\ell)J_\ell(k\ell).$$

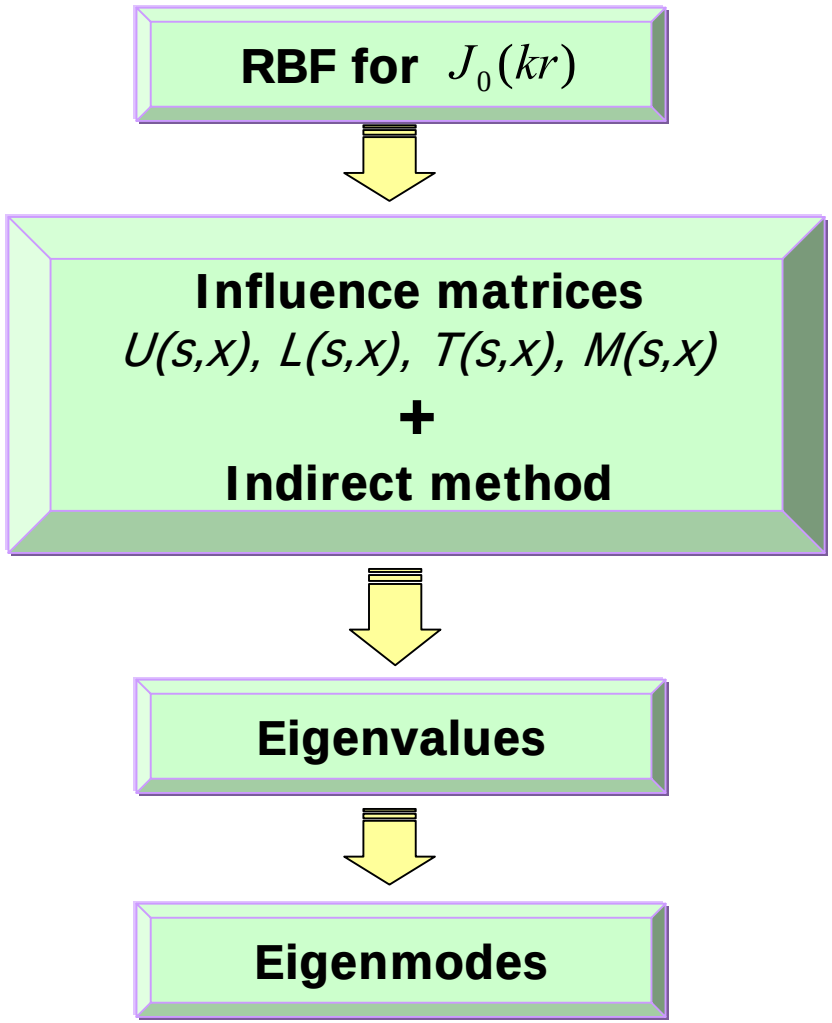
$$0 = J_0(k\ell)J'_0(k\ell) + 2\sum_{m=-\infty}^{\infty} J_m(k\ell)J'_m(k\ell).$$

Addition theorem of Bessel function

$diag\,([T(s,x)])$

$$b_0 = 0.$$

Two-dimensional acoustic eigenproblem



SVD (Singular Value Decomposition)

$$\left[A(\lambda)\right]\{x\}=0$$

SVD

$$\left[A\right]_{m\times n}=\left[\Phi\right]_{m\times m}\left[\Sigma\right]_{m\times n}\left[\Psi\right]^{T}_{n\times n}$$

J. T. Chen, C. F. Lee and S. Y. Lin.

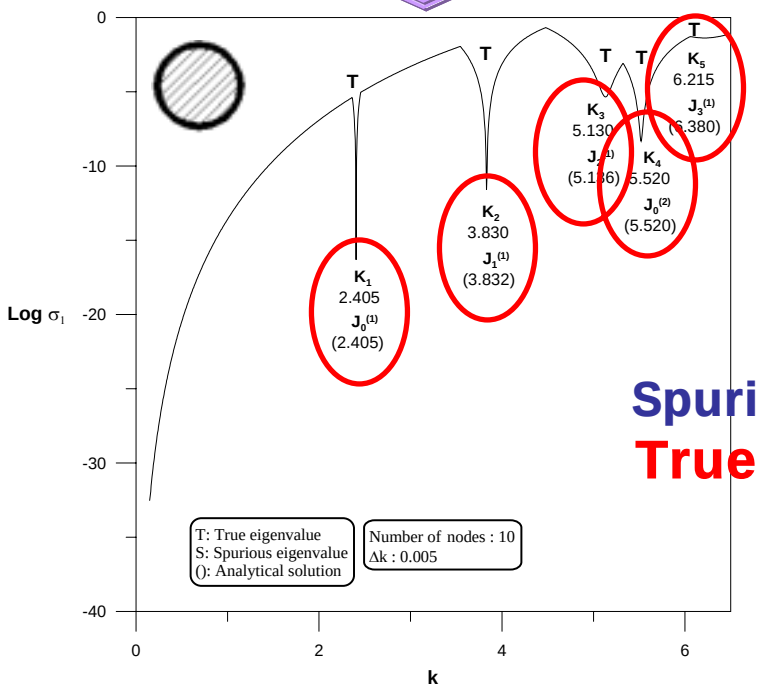
A new point of view for the polar decomposition using singular value decomposition.

International Journal of Computational and Numerical Analysis and Applications.

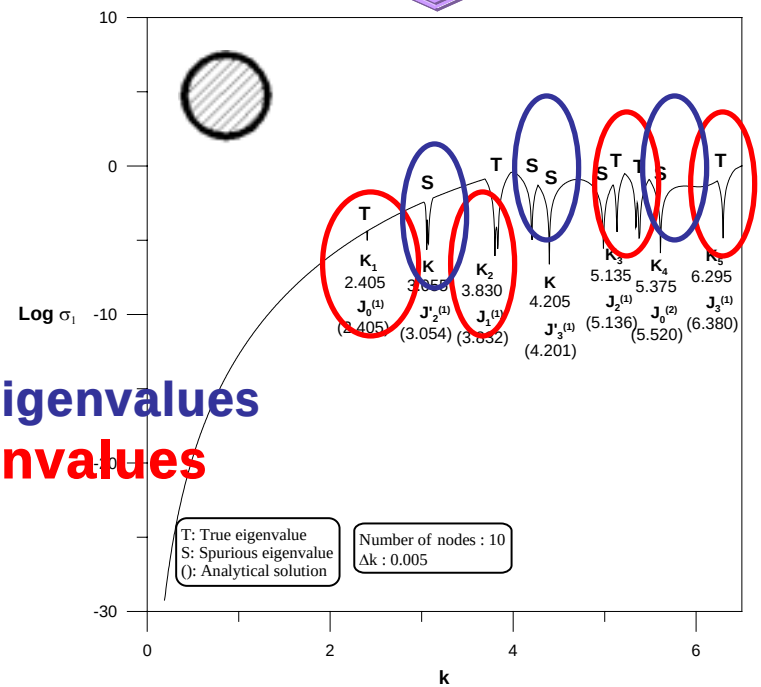
Vol.2, No.3, 257-264 , 2002.

Eigenfrequencies for Dirichlet B.C.

Single-layer potential approach



Double-layer potential approach

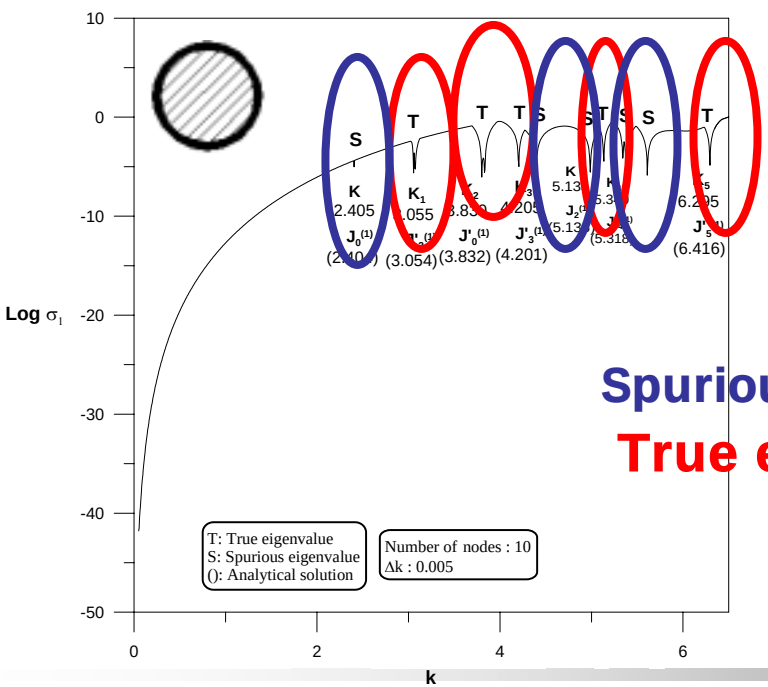


Spurious eigenvalues

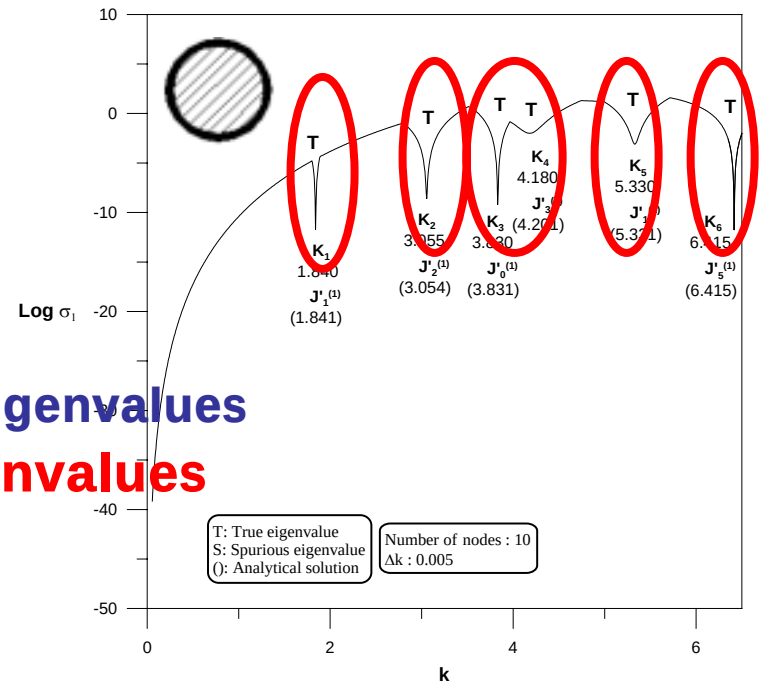
True eigenvalues

Eigenfrequencies for Neumann B.C.

Single-layer potential approach



Double-layer potential approach



Spurious eigenvalues

True eigenvalues

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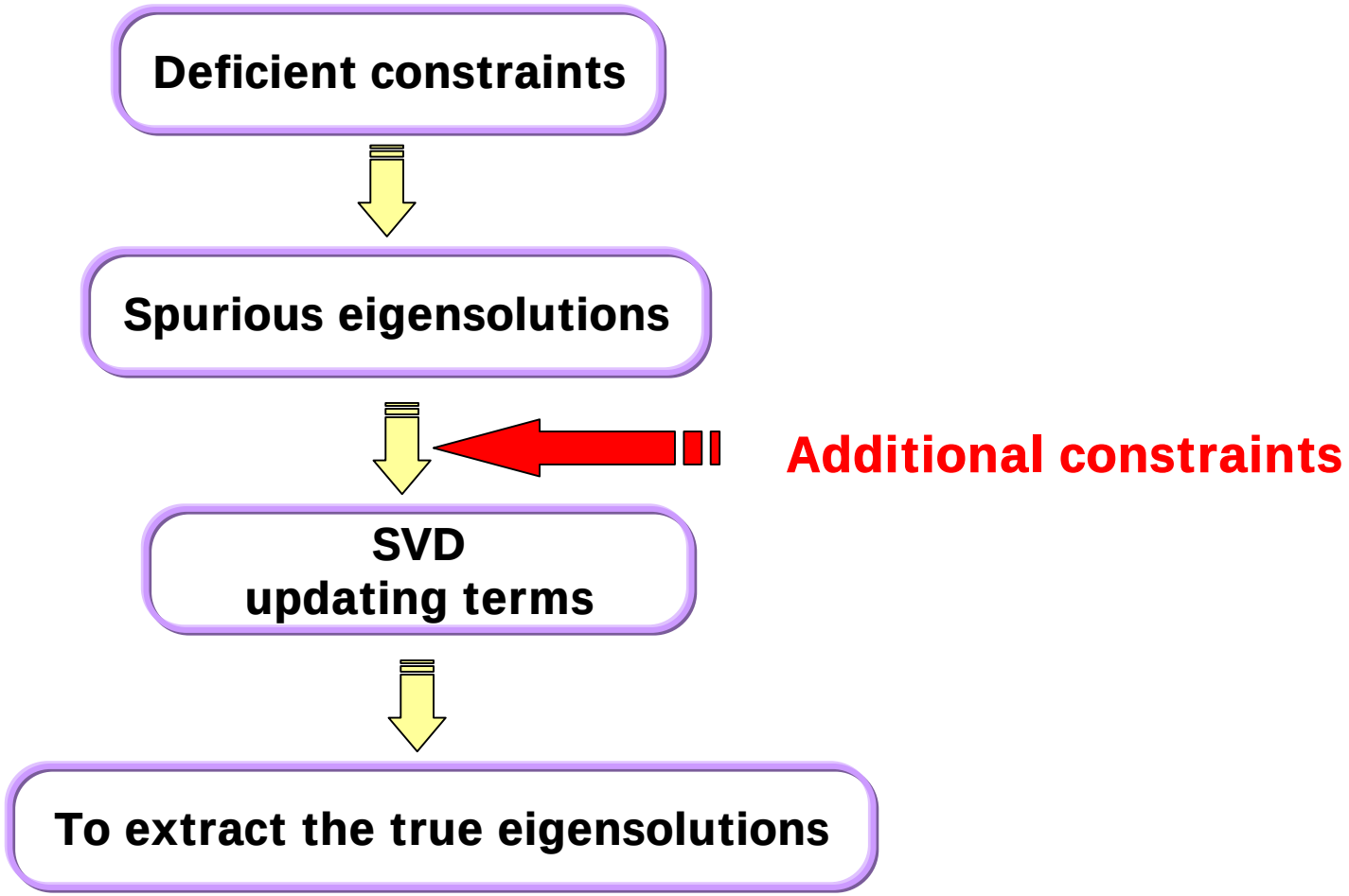
SVD

Diagonal matrix

$$[\Sigma]=\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times n}$$

Unitary matrix $[\Phi]_{m\times m}$, $[\Psi]_{n\times n}$

To extract the true eigenvalues



SVD updating terms

Direct method for Dirichlet B. C. :

Singular equation (UT method)

$$[T^E]\,\underset{\sim}{u} = [U^E]\,\underset{\sim}{t} = 0.$$

Hypersingular equation (LM method)

$$[M^E]\,\underset{\sim}{u} = [L^E]\,\underset{\sim}{t} = 0.$$

$$\underset{\sim}{t} = \{\psi_j\}$$

$$\begin{bmatrix} U^E \\ L^E \end{bmatrix}\{\psi_j\} = 0$$

SVD updating terms

To extract the true eigenfrequencies

SVD updating terms

$$[U^E]\{\psi_i\} = \{0\}$$

$$[L^E]\{\psi_i\} = \{0\}$$

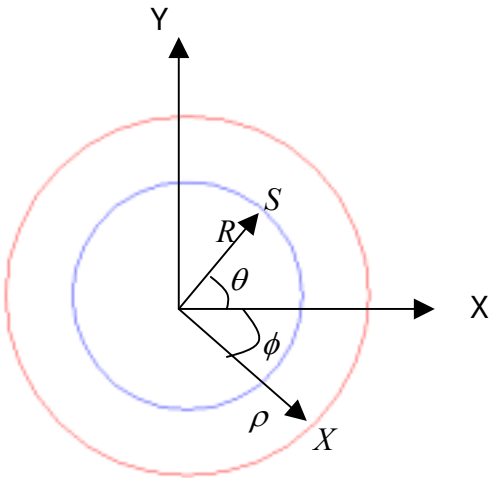
$$\sum_j \sigma_j^U \{\phi_j^U\}\{\psi_j^U\}^T \{\psi_i\} = \{0\} \xrightarrow[\psi_j^U \cdot \psi_j = \delta_{ij}]{} \sigma_j^U \{\phi_j^U\} = \{0\}$$

$$\sum_j \sigma_j^L \{\phi_j^L\}\{\psi_j^L\}^T \{\psi_i\} = \{0\} \xrightarrow[\psi_j^L \cdot \psi_j = \delta_{ij}]{} \sigma_j^L \{\phi_j^L\} = \{0\}$$

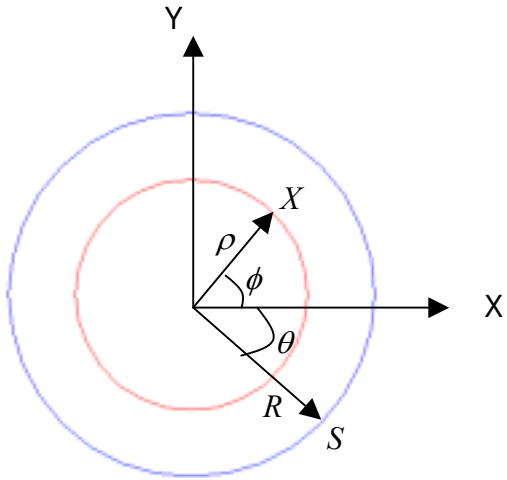
$$\sigma_i^U = \sigma_i^L = 0$$

The degenerate kernels

The degenerate kernels for interior and exterior problems:



Interior problem



Exterior problem

Blue: field points
Red: source points

$$U(s,x)=\begin{cases} U^{I}(R,\theta\,;\,\rho,\phi)=-\dfrac{\pi}{2}\displaystyle\sum_{n=-\infty}^{\infty}J_m(kR)J_m(k\rho)\big(\cos(m(\theta-\phi))\big), & R>\rho \\ U^{E}(R,\theta\,;\,\rho,\phi)=-\dfrac{\pi}{2}\displaystyle\sum_{n=-\infty}^{\infty}J_m(k\rho)J_m(kR)\big(\cos(m(\theta-\phi))\big), & R<\rho \end{cases}$$

$$U^{E}=U^{I}$$

The degenerate kernels

$$T(s,x)=\begin{cases} T^{I}(R,\theta;\rho,\phi)=-\dfrac{k\pi}{2}\displaystyle\sum_{n=-\infty}^{\infty}J_m(k\rho)J'_m(kR)\big(\cos(m(\theta-\phi))\big), & R>\rho \\ T^{E}(R,\theta;\rho,\phi)=-\dfrac{k\pi}{2}\displaystyle\sum_{n=-\infty}^{\infty}J_m(kR)J'_m(k\rho)\big(\cos(m(\theta-\phi))\big), & R<\rho \end{cases}$$

$$L^{E}=T^{I}$$

$$L(s,x)=\begin{cases} L^{I}(R,\theta;\rho,\phi)=-\dfrac{k\pi}{2}\displaystyle\sum_{n=-\infty}^{\infty}J'_m(k\rho)J_m(kR)\big(\cos(m(\theta-\phi))\big), & R>\rho \\ L^{E}(R,\theta;\rho,\phi)=-\dfrac{k\pi}{2}\displaystyle\sum_{n=-\infty}^{\infty}J'_m(kR)J_m(k\rho)\big(\cos(m(\theta-\phi))\big), & R<\rho \end{cases}$$

$$\begin{bmatrix} U^{E} \\ L^{E} \end{bmatrix} \xrightarrow[L^{E}=T^{I}]{U^{E}=U^{I}} \begin{bmatrix} U^{I} \\ T^{I} \end{bmatrix}$$

SVD techniques for eigensolutions

Boundary Value Problem	Eigen-solution	Density function		True and spurious eigenvalues			
		Single-layer potential	Double layer potential	Direct method		Indirect method	
Dirichlet Problem	True	$J_m(k\rho)=0$	$J_m(k\rho)=0$	SVD updating term	$\begin{bmatrix} U^E \\ L^E \end{bmatrix}$	SVD updating term	$\begin{bmatrix} U^I \\ T^I \end{bmatrix}$
	Spurious	$J'_m(k\rho)=0$	$J_m(k\rho)=0$	SVD updating document	$\begin{bmatrix} L^E & M^E \end{bmatrix}$	SVD updating document	$\begin{bmatrix} T^I & M^I \end{bmatrix}$
Neumann Problem	True	$J'_m(k\rho)=0$	$J'_m(k\rho)=0$	SVD Updating term	$\begin{bmatrix} T^E \\ M^E \end{bmatrix}$	SVD updating term	$\begin{bmatrix} L^I \\ M^I \end{bmatrix}$
	Spurious	$J_m(k\rho)=0$	$J'_m(k\rho)=0$	SVD updating document	$\begin{bmatrix} U^E & T^E \end{bmatrix}$	SVD updating document	$\begin{bmatrix} U^I & L^I \end{bmatrix}$

To extract the true eigenvalues by the indirect method

Indirect method for Dirichlet B. C. :

$$\begin{bmatrix} U^I \\ T^I \end{bmatrix} \{\psi_j\} = 0$$

SVD updating terms

$$\sum_j \sigma_j^U \{\phi_j^U\}\{\psi_j^U\}^T \{\psi_i\} = \{0\} \xrightarrow[\psi_j^U \cdot \psi_j = \delta_{ij}]{} \sigma_j^U \{\phi_j^U\} = \{0\}$$

$$\sum_j \sigma_j^T \{\phi_j^T\}\{\psi_j^T\}^T \{\psi_i\} = \{0\} \xrightarrow[\psi_j^T \cdot \psi_j = \delta_{ij}]{} \sigma_j^T \{\phi_j^T\} = \{0\}$$

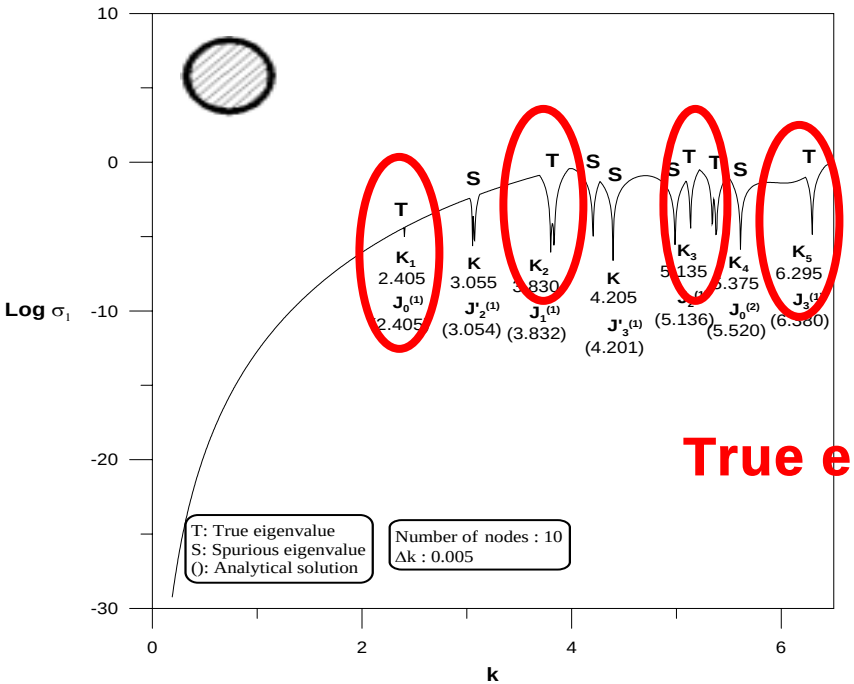
Singular-layer approach: $J_i(k\rho) J_i(k\rho) = 0, \quad (i \text{ no sum})$

Double-layer approach: $J_i(k\rho) J_i'(k\rho) = 0, \quad (i \text{ no sum})$

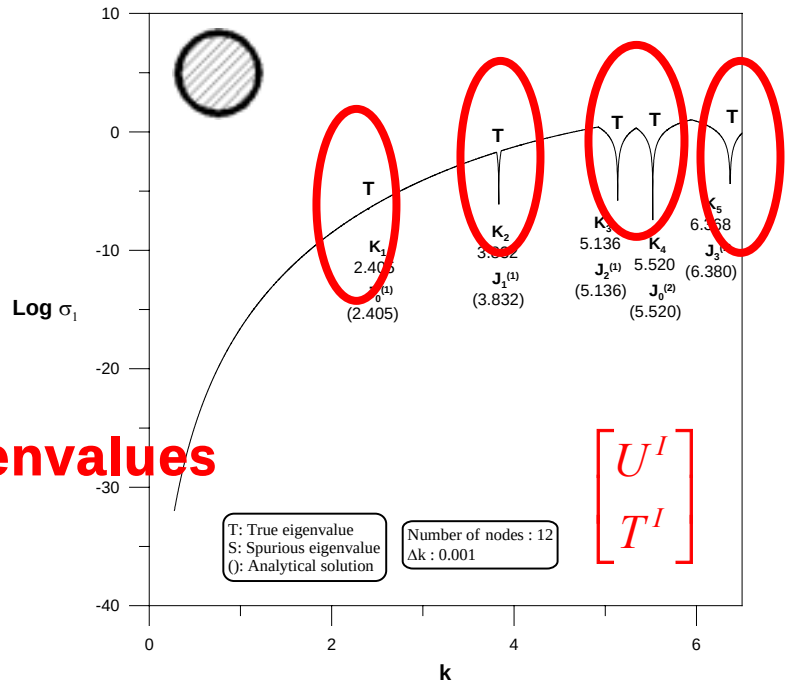
Examples for the SVD updating terms

For Dirichlet B.C.

Double-layer potential approach



SVD updating-terms



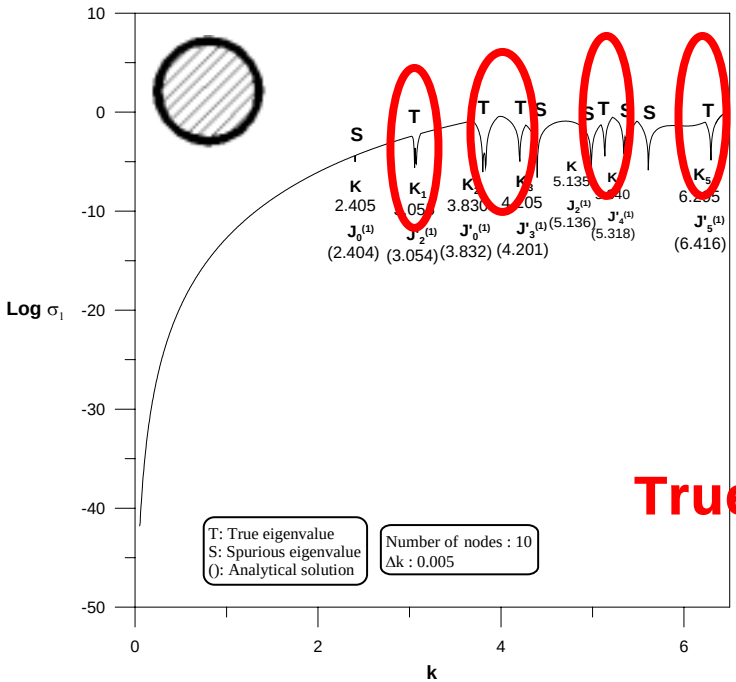
True eigenvalues

$$\begin{bmatrix} U^I \\ T^I \end{bmatrix}$$

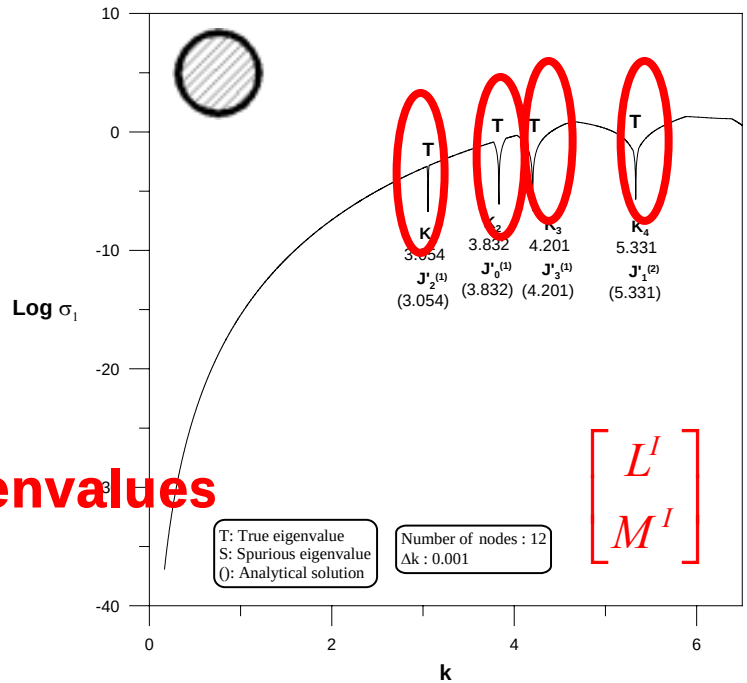
Examples for the SVD updating terms

For Neumann B.C.

Single-layer potential approach

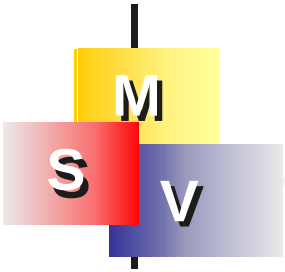


SVD updating-terms



True eigenvalues

$$\begin{bmatrix} L^I \\ M^I \end{bmatrix}$$



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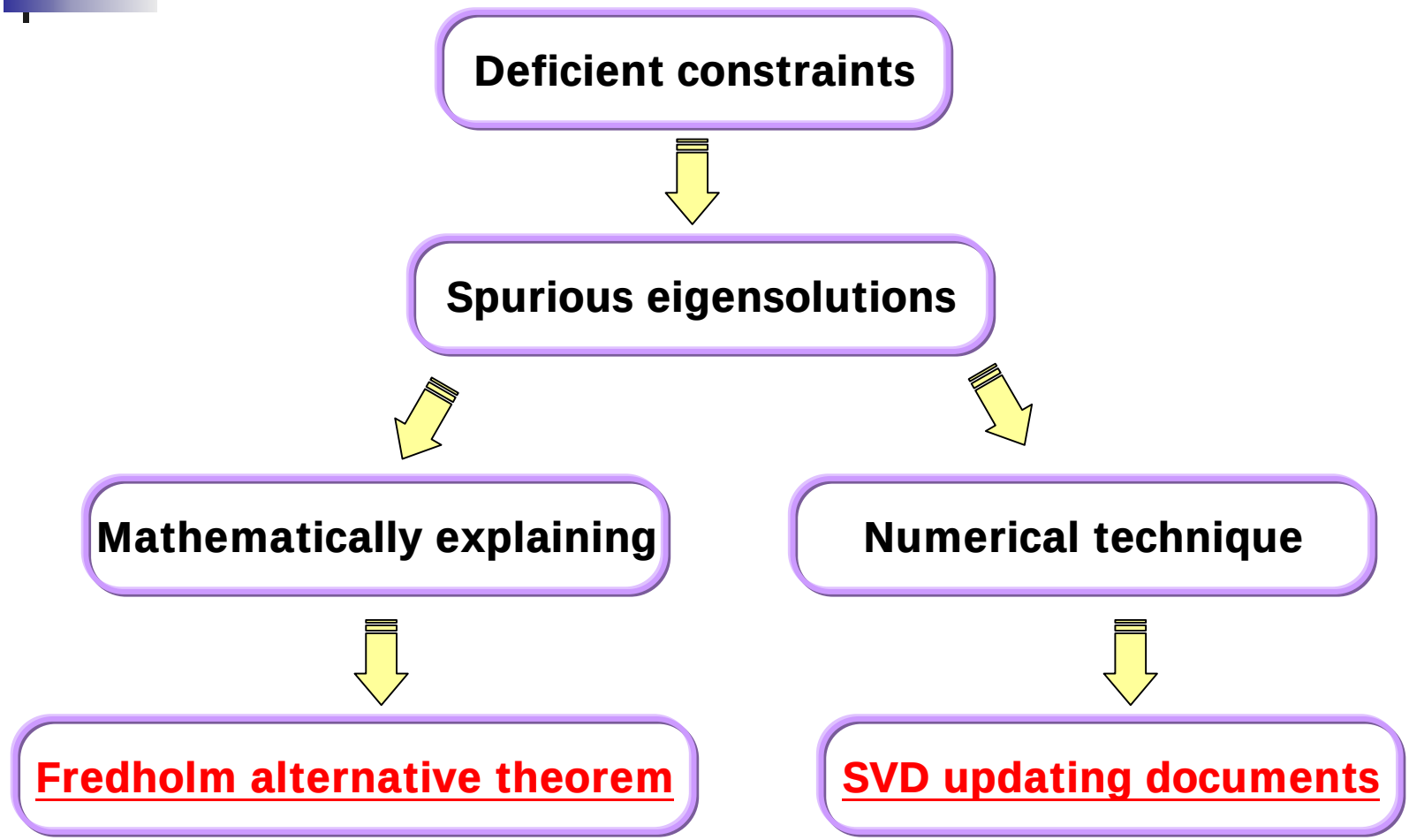
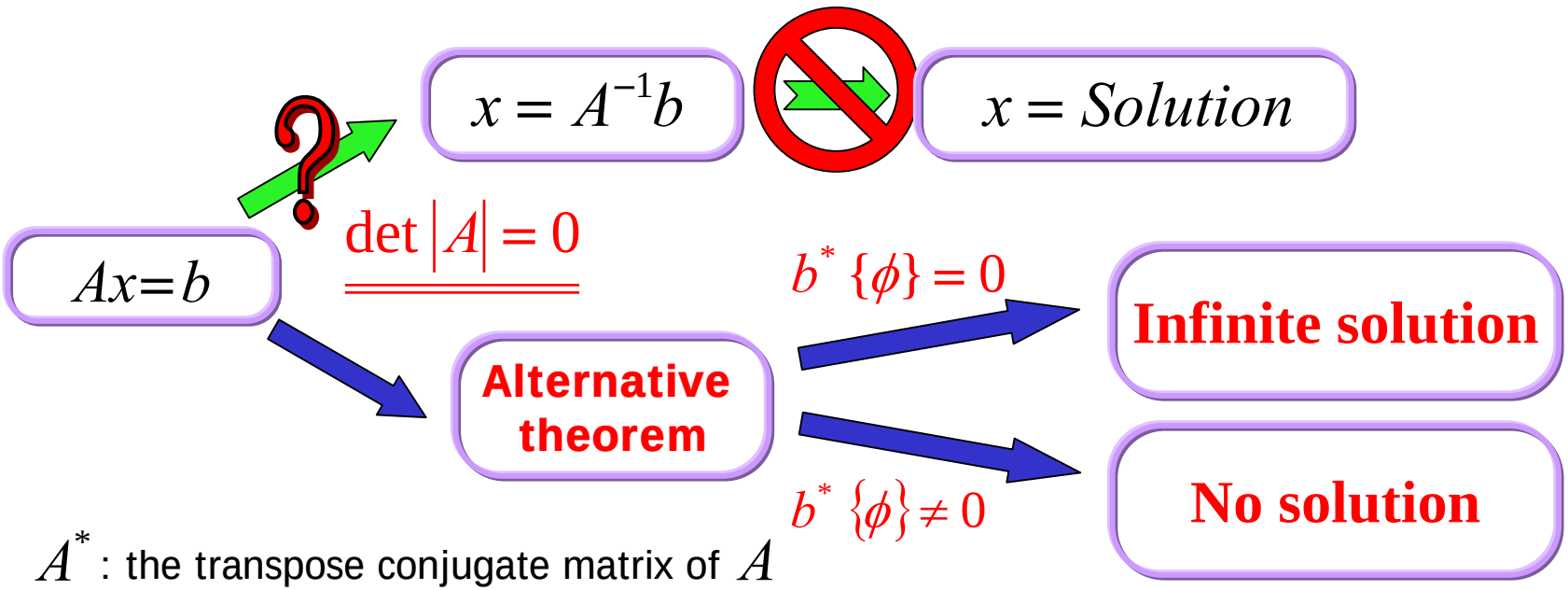
To filter out the spurious solutions



Fredholm alternative theorem

Fredholm's alternative theorem:

For solving an algebraic system: $Ax = b$



A^* : the transpose conjugate matrix of A

$A^* = A^T$ if A is real

where f satisfies $A^*\{\phi\} = 0$

SVD updating documents

For double-layer potential approach:

$$\begin{Bmatrix} u(x) \\ t(x) \end{Bmatrix} = \begin{bmatrix} T(s,x) \\ M(s,x) \end{bmatrix} \{\psi\}$$

b A x

$$A^T\{\phi\} = 0 \quad \text{or} \quad \{\phi\}^T A = 0$$

$$\begin{bmatrix} [T]^T \\ [M]^T \end{bmatrix} \{\phi\} = 0 \quad \text{or} \quad \{\phi\}^T \begin{bmatrix} [T] & [M] \end{bmatrix} = 0$$

SVD updating documents

For double-layer potential approach:

SVD updating documents

$$\sum_j \sigma_j^T \{\psi_j^T\}\{\phi_j^T\}^T\{\phi_i\} = \{0\} \xrightarrow[\{\phi_j^T\}^T\cdot\phi_j=\delta_{ij}]{} \sigma_j^T\{\psi_j^T\} = \{0\}$$

$$\sum_j \sigma_j^M \{\psi_j^M\}\{\phi_j^M\}^T\{\phi_i\} = \{0\} \xrightarrow[\{\phi_j^M\}^T\cdot\phi_j=\delta_{ij}]{} \sigma_j^M\{\psi_j^M\} = \{0\}$$

$$\Phi\Sigma\Psi^T$$

$$\sigma_i^T = \sigma_i^M = 0$$

Spurious modes

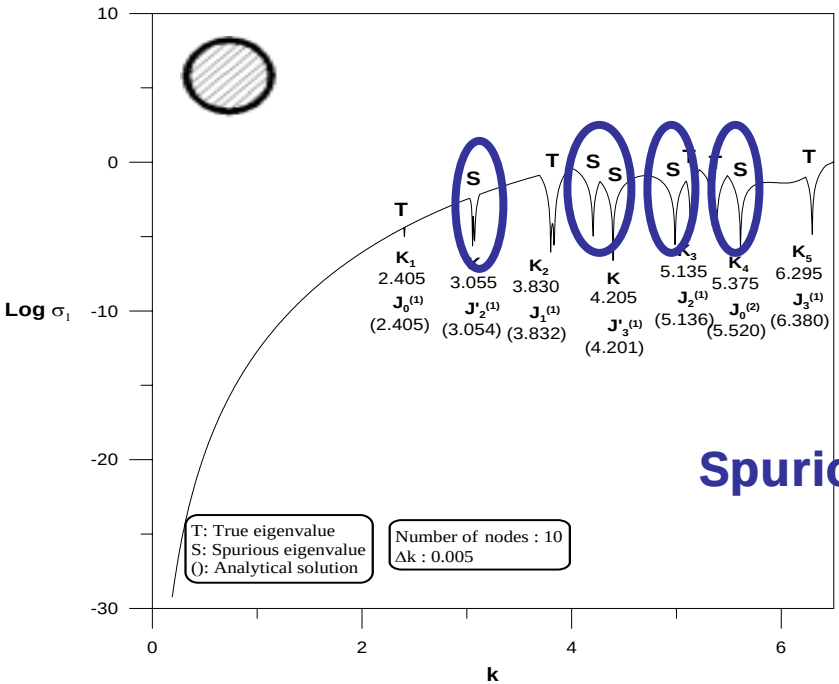
Dirichlet problem: $J_i(k\rho)J_i'(k\rho) = 0, \quad (i \text{ no sum})$

Neumann problem: $J_i'(k\rho)J_i'(k\rho) = 0, \quad (i \text{ no sum})$

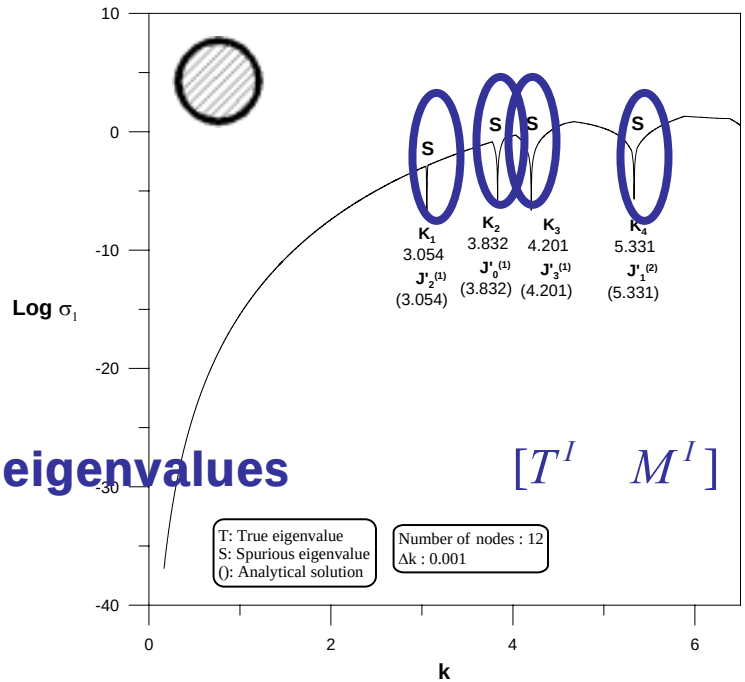
Examples for the SVD updating documents

For Dirichlet B.C.

Double-layer potential approach



SVD updating documents

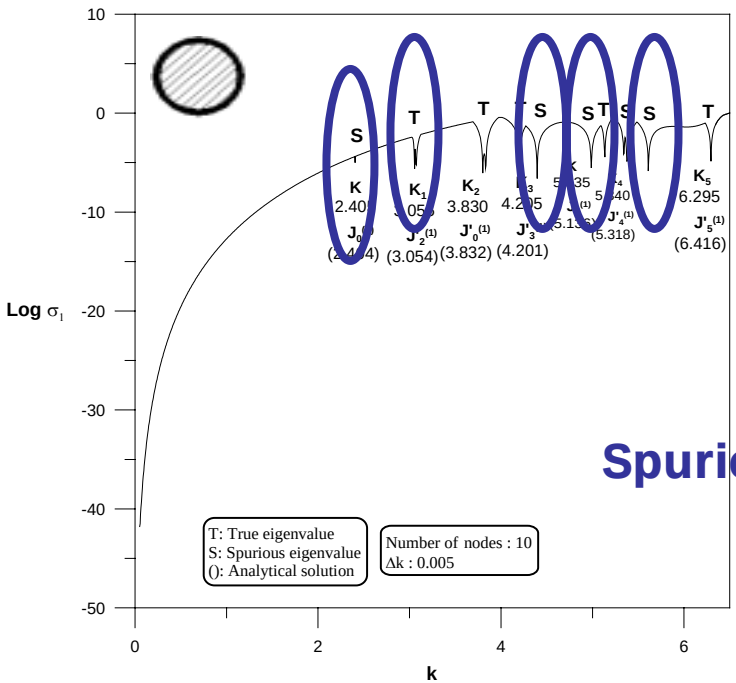


Spurious eigenvalues $[T^I \quad M^I]$

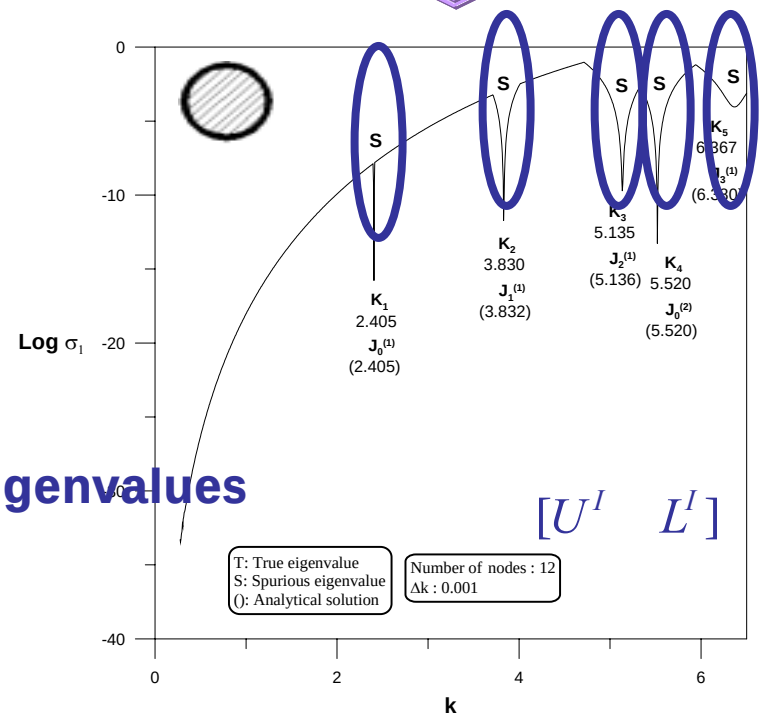
Examples for the SVD updating documents

For Dirichlet B.C.

Double-layer potential approach



SVD updating documents

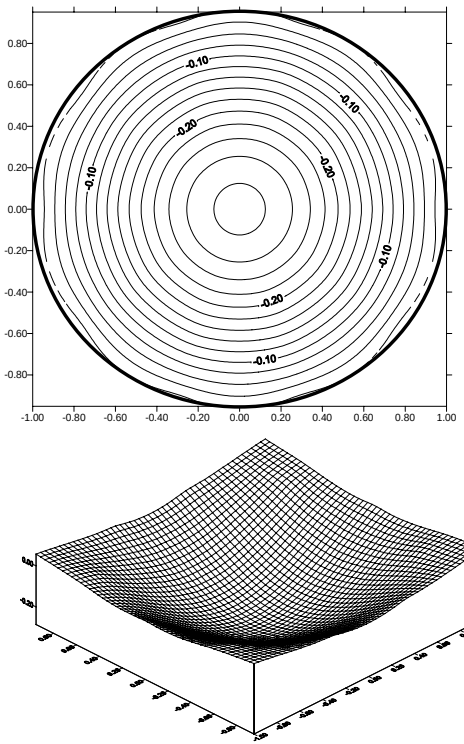


Spurious eigenvalues

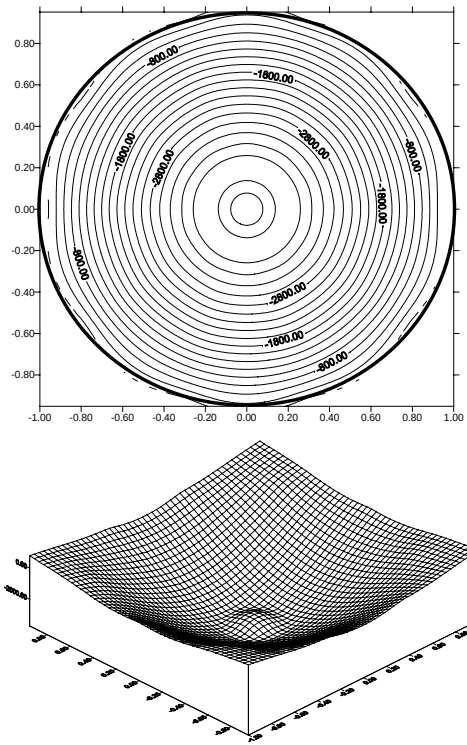
$$[U^I \quad L^I]$$

The interior modes of a circular acoustics

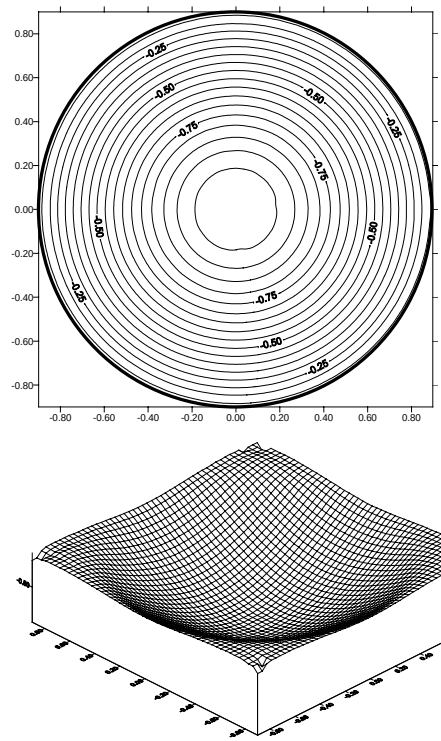
Single-layer protential approach



Double-layer protential approach



Analytical solution



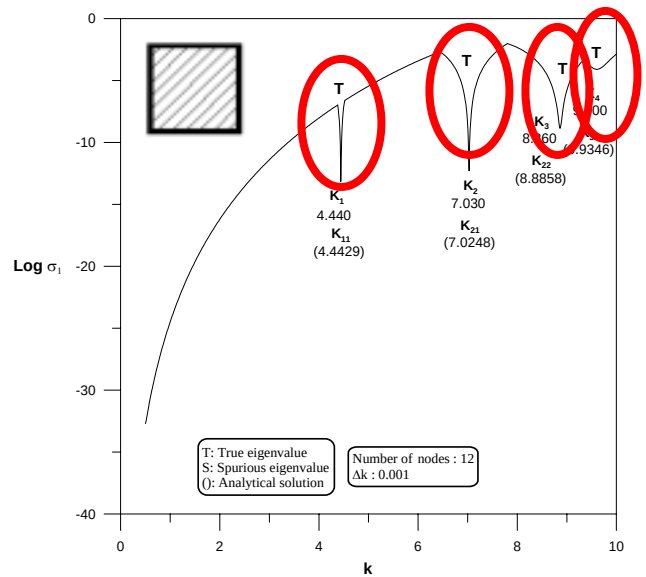
SVD techniques for a square acoustics

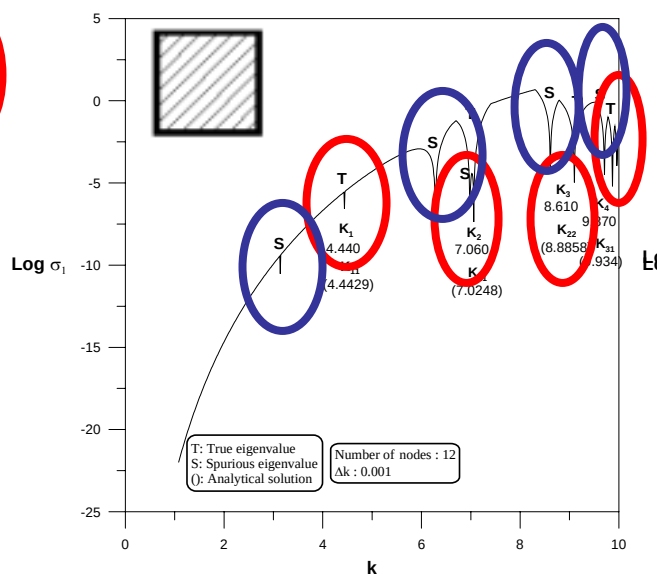
For Dirichlet B.C.

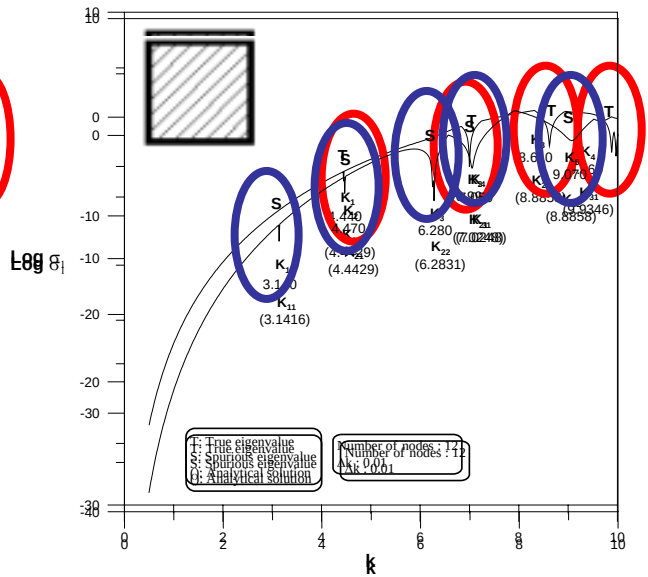
Single-layer protential approach

Double-layer protential approach

SVD updating documents



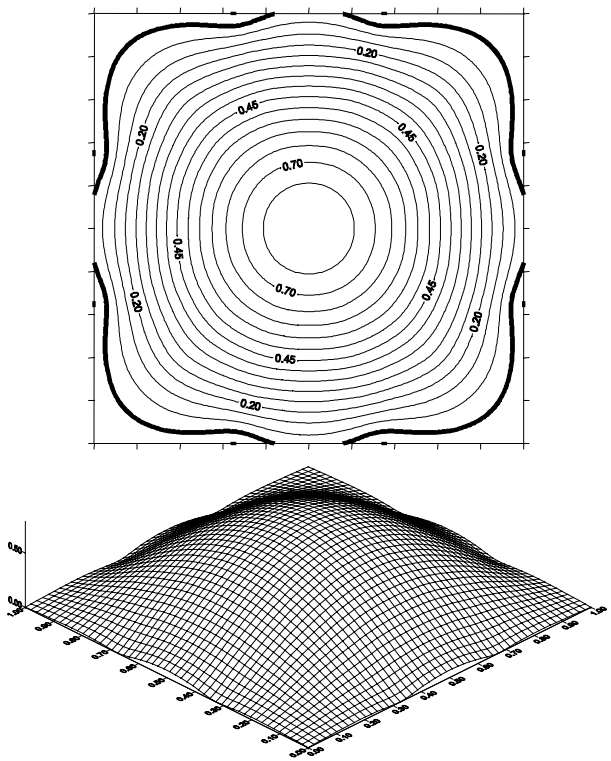




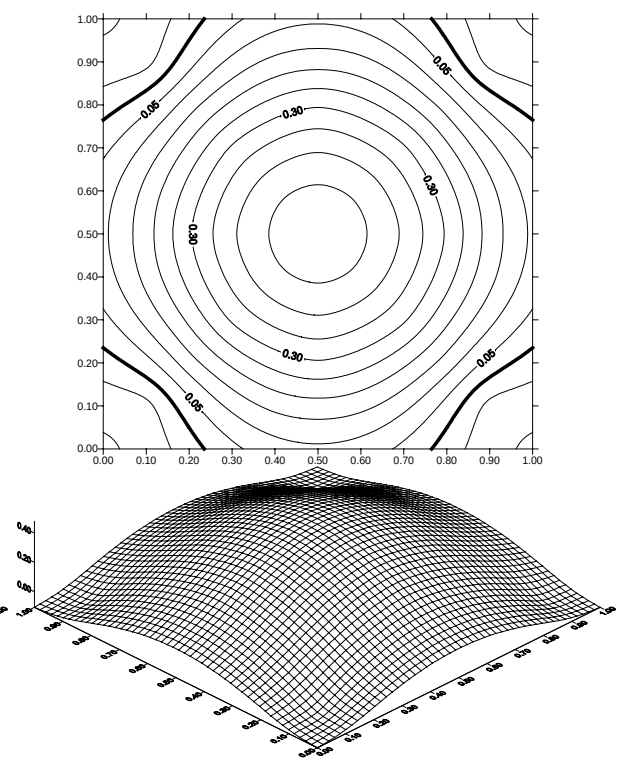
True eigenvalues Spurious eigenvalues

The interior modes of square acoustics

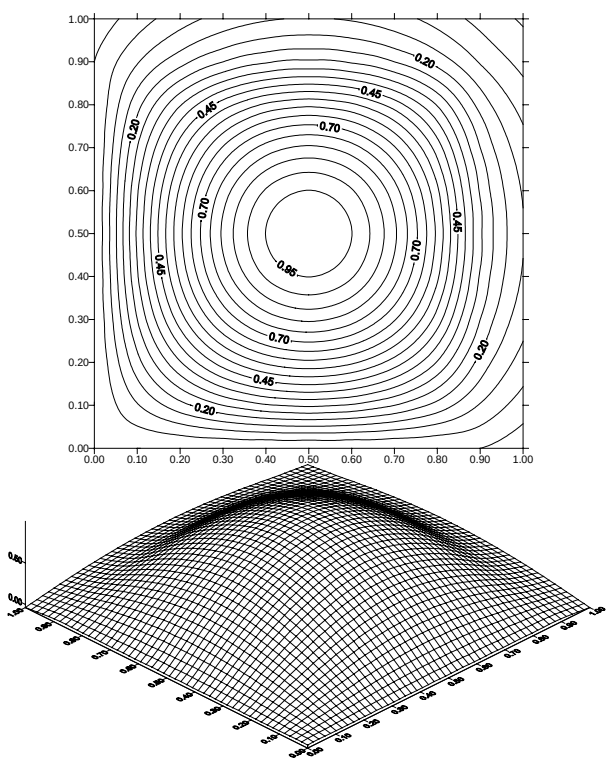
Single-layer protential approach



Double-layer protential approach



Analytical solution



Outlines

1. Introduction
2. 2-D acoustic eigenproblem
3. SVD updating terms
4. SVD updating documents
5. 3-D cavity
6. Conclusions

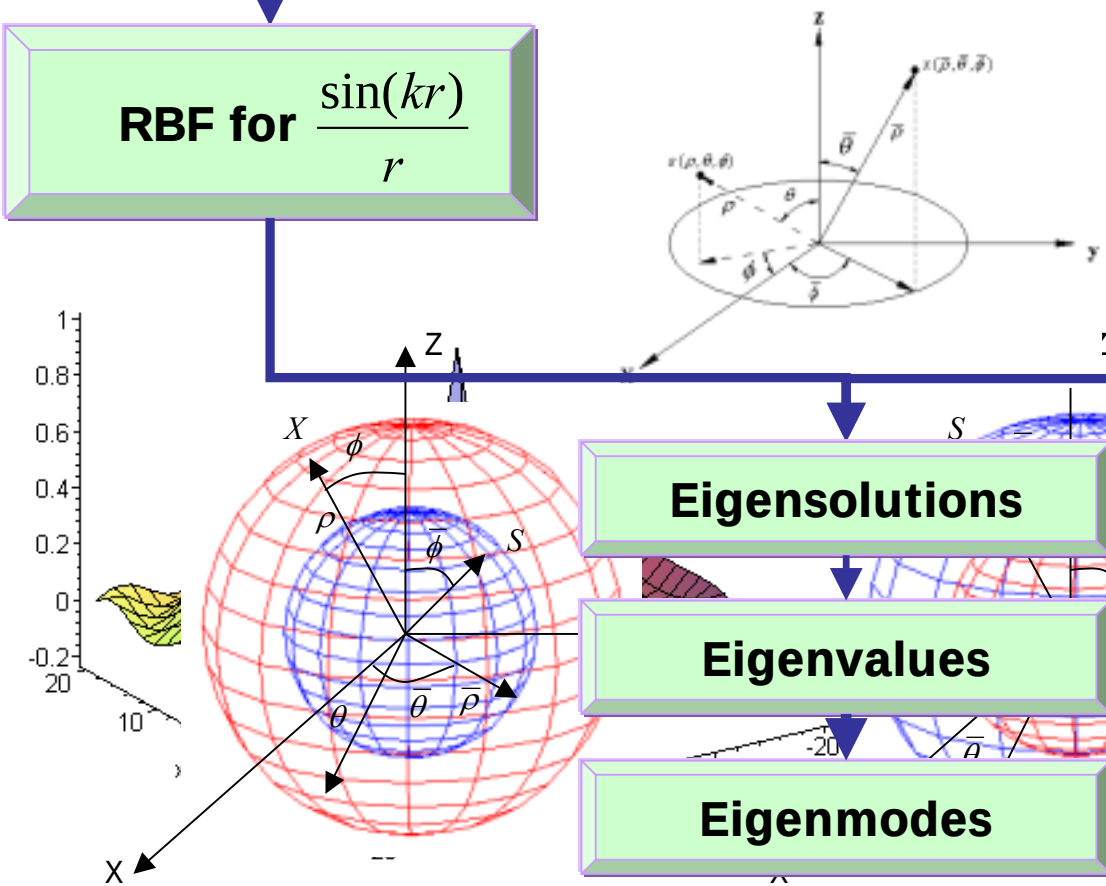
Three-dimensional spherical cavity

Indirect method

RBF for $\dfrac{\sin(kr)}{r}$

Continuous system

Degenerate kernel



Eigensolutions

Eigenvalues

Eigenmodes

Blue：source points
Red：field points

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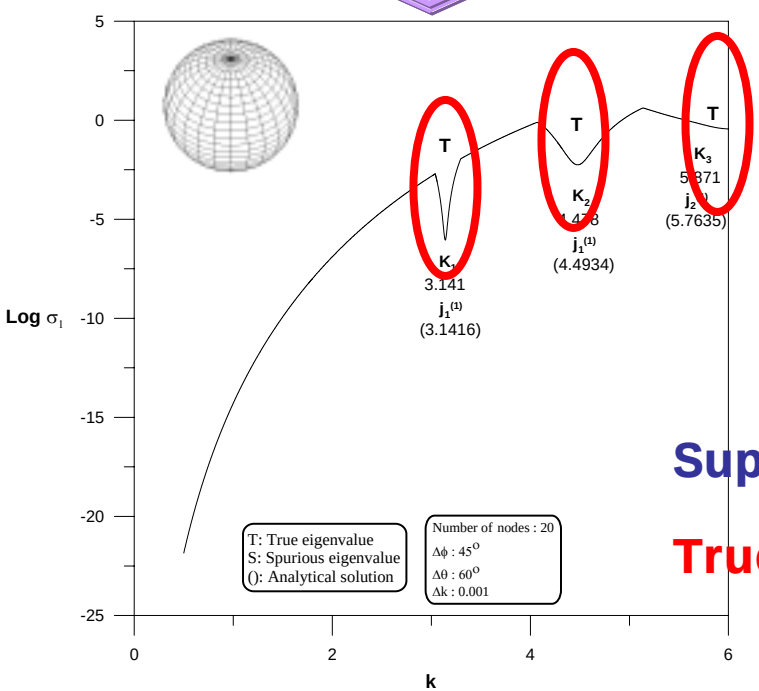
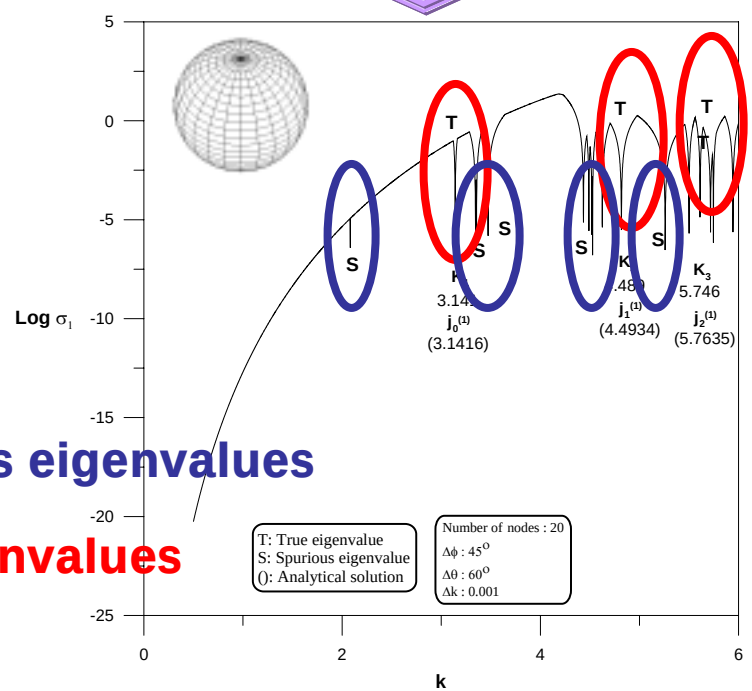
Eigenfrequencies for Dirichlet B.C.

Single-layer potential approach



Double-layer potential approach

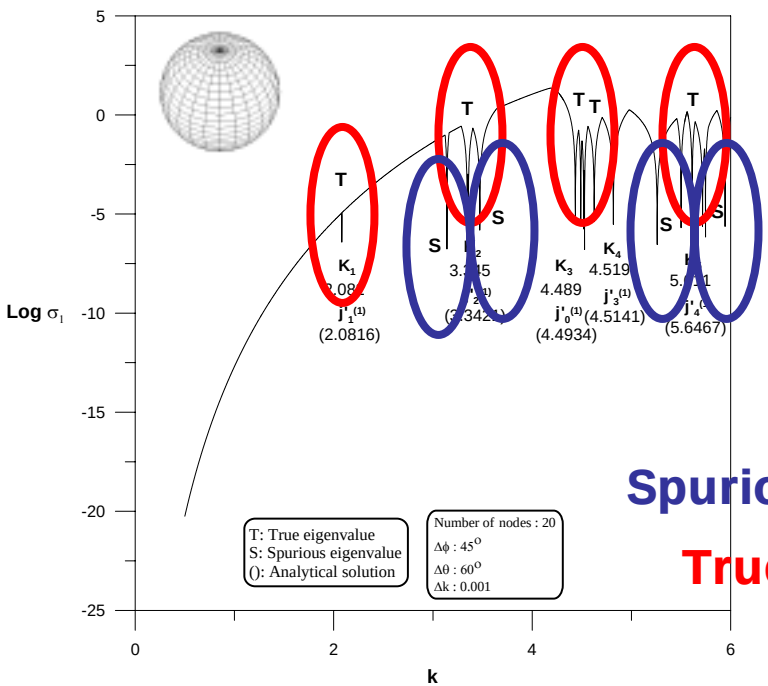


Supurious eigenvalues

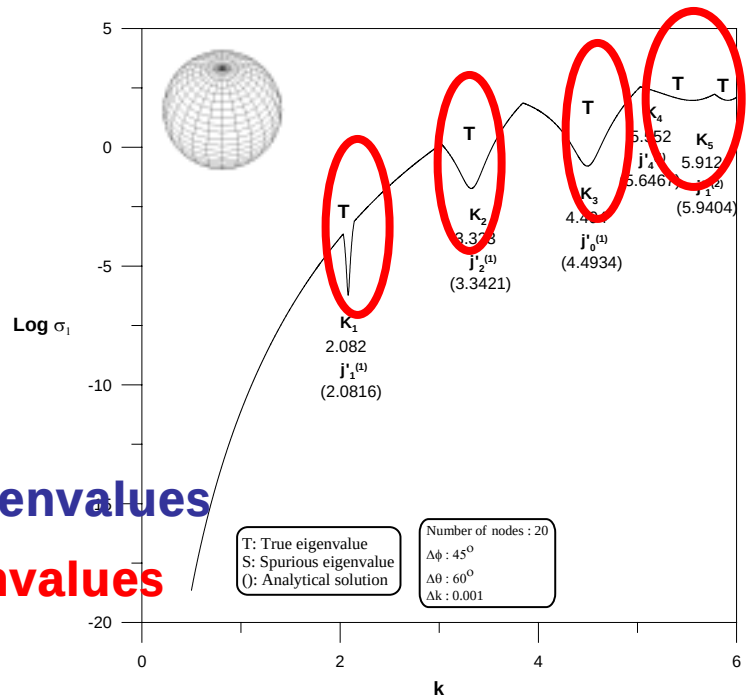
True eigenvalues

Eigenfrequencies for Neumann B.C.

Single-layer potential approach



Double-layer potential approach



Spurious eigenvalues

True eigenvalues

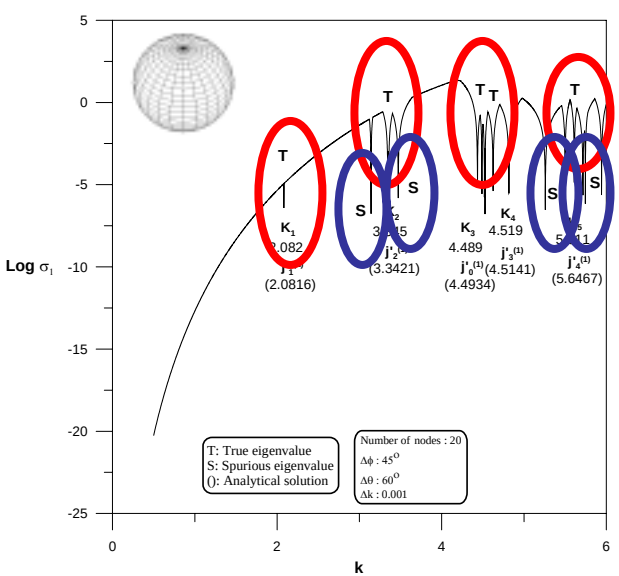
SVD techniques for a spherical cavity

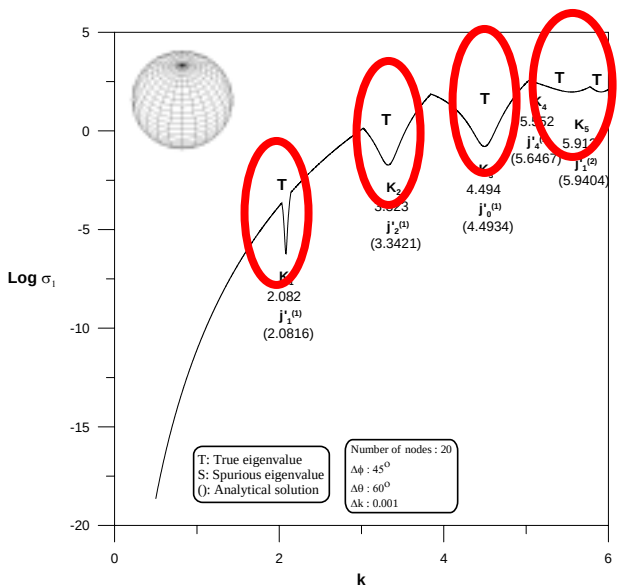
Neumann B.C.

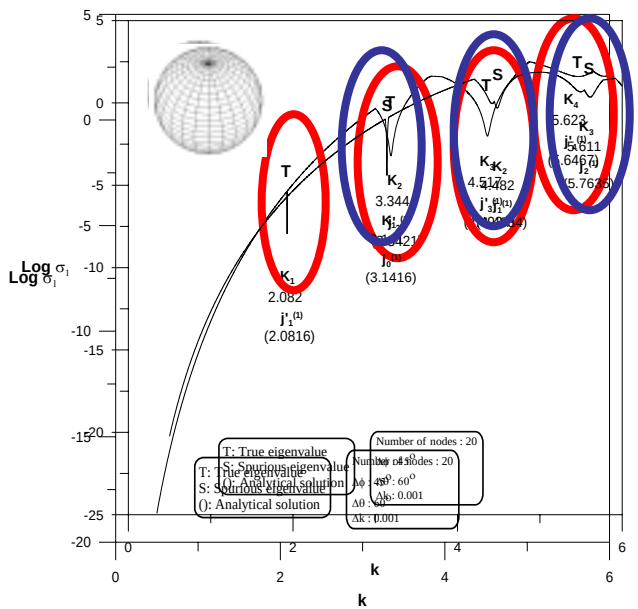
Single-layer protential approach

Double-layer protential approach

SVD updating documents







Spurious eigenvalues

True eigenvalues

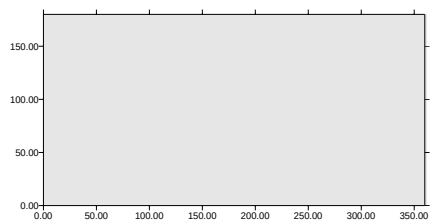
Contours for the boundary modes

$$u(1,\theta,\phi)=\begin{cases}1, & n=m=0\\ p_n^m(\cos\theta)e^{im\phi}, & \text{otherwise}\end{cases}$$

n =0, k=0.0

m =0

Analytical modes



Single-layer approach

Double-layer approach

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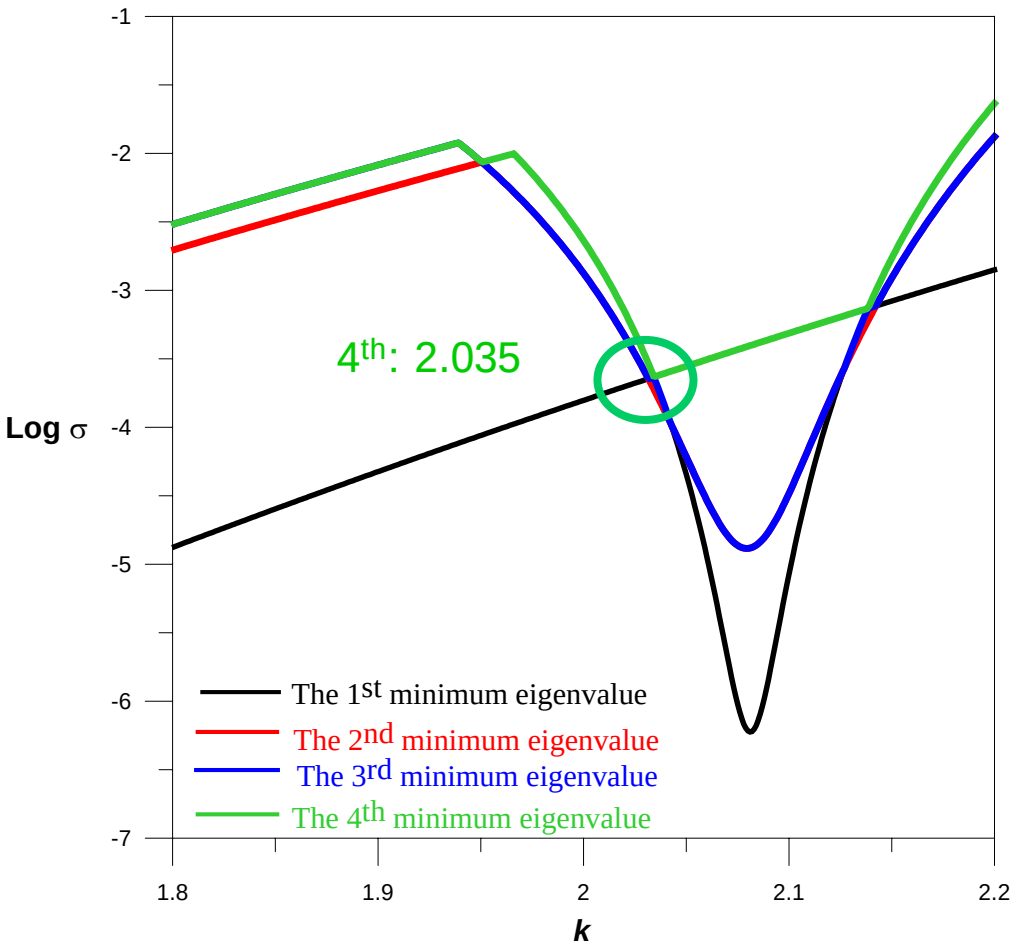
Multiplicity of eigenvalues



i^{th} minimum eigenvalue	Analytical eigenvalue	Numerical eigenvalue	Test
1st	2.082	2.081	∨
2nd		2.079	∨
3rd		2.079	∨
4th		2.035	×

Multiplicity=3

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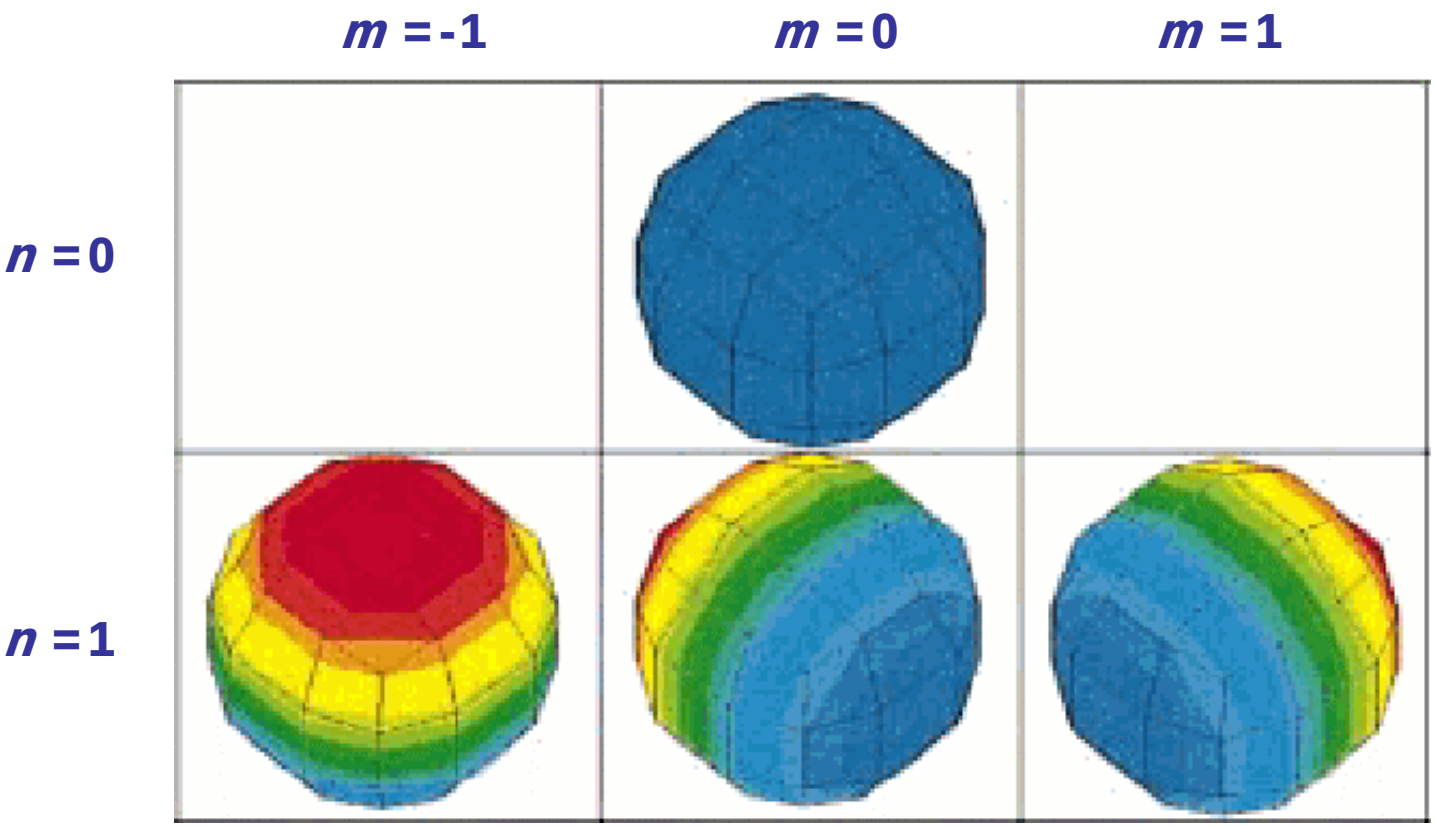
Spherical harmonics

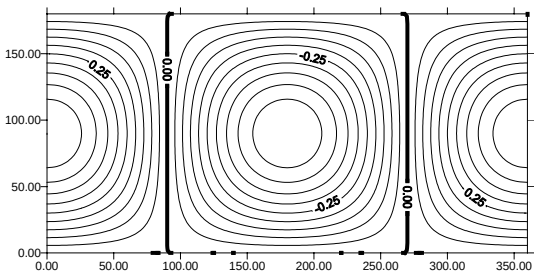
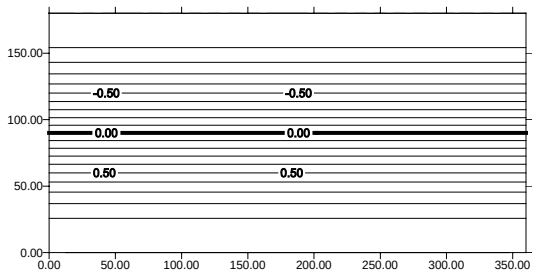
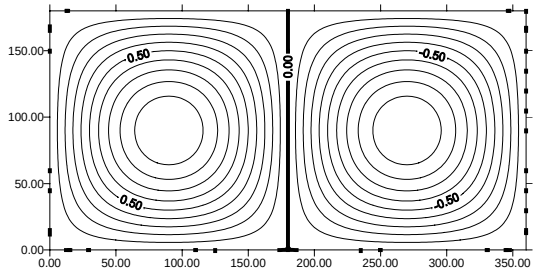
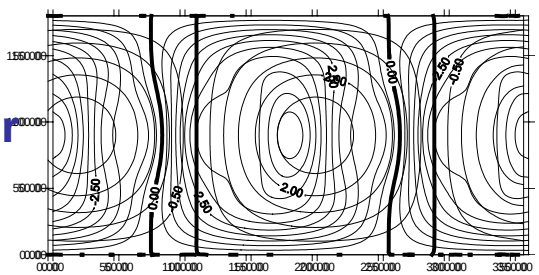
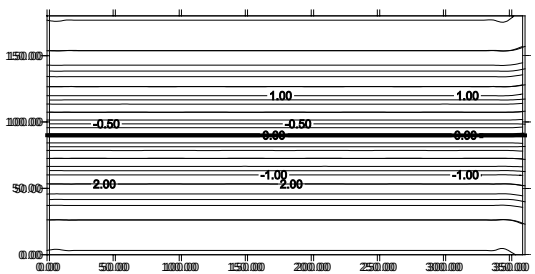
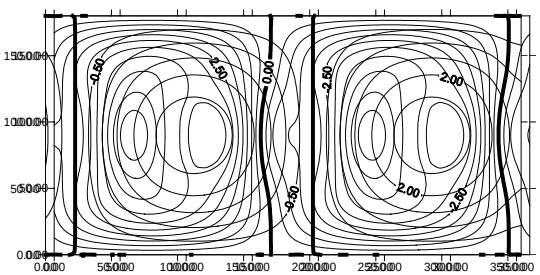
$$u(1,\theta,\phi)=\begin{cases}1, & n=m=0\\ p_n^m(\cos\theta)e^{im\phi}, & \text{otherwise}\end{cases}$$



P. A. NELSON
Y. KAHANA

Contours of the boundary modes

$n = 1,\ k = 2.082$

	$m = -1$	$m = 0$	$m = -1$
Analytical modes			
Single layer approach Double layer approach			

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Outlines

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6. Conclusions

Conclusions

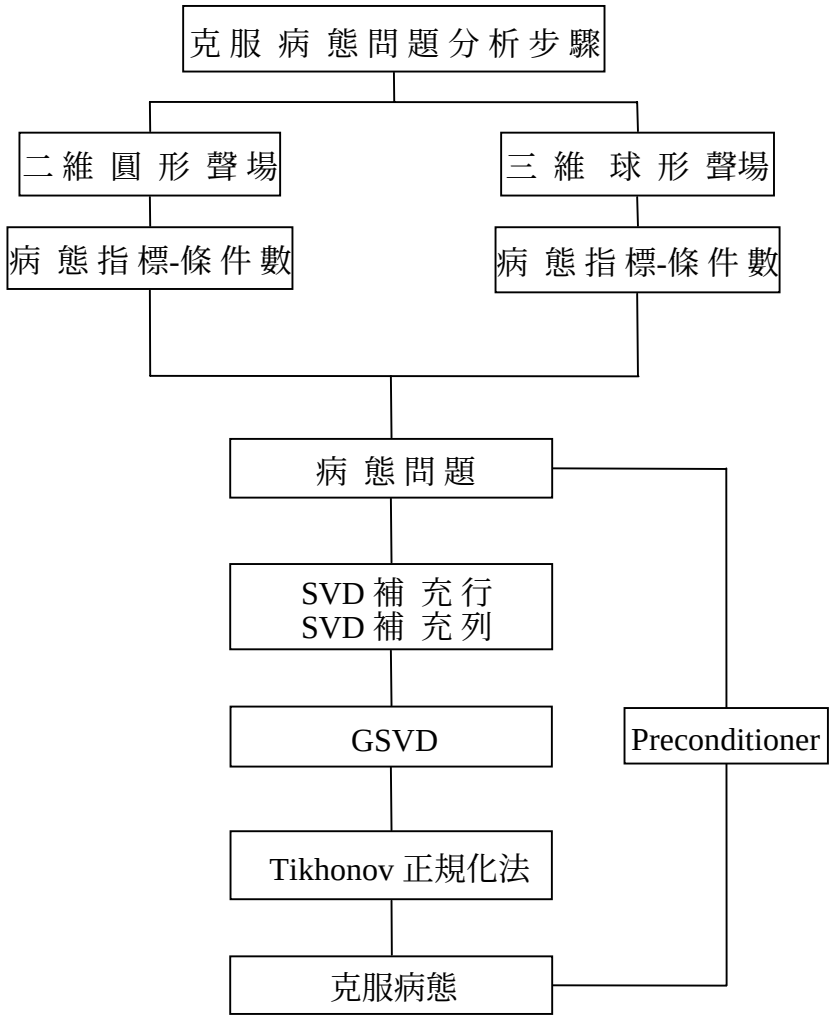
I. Only the boundary nodes are required such that the influence matrices can be determinated by the two-point function.

II. The diagonal elements can be derived by invariant method.

III. Based on the imaginary-part formulations, the deficient constraints cause the spurious eigensolutions.

IV. The SVD techniques can solve the true or spurious eigensolutions well.

To overcome the ill-posed problems



Distributed & concentrated type

	Distributed-type	Concentrated-type*
Single-layer potential approach	Dirichlet problem $u(x)=\int_B U^I(s,x)\phi(s)dB(s)$ Neumann problem $t(x)=\int_B L^I(s,x)\phi(s)dB(s)$	Dirichlet problrm $u(x_i)=\sum_j U^I(s_j,x_i)A_j=(SM)_{ij}A_j$ Neumann problrm $t(x_i)=\sum_j L^I(s_j,x_i)B_j=(SM_x)_{ij}B_j$
Double-layer potential approach	Dirichlet problrm $u(x)=\int_B T^I(s,x)\psi(s)dB(s)$ Neumann problrm $t(x)=\int_B M^I(s,x)\psi(s)dB(s)$	Dirichlet problrm $u(x_i)=\sum_j T^I(s_j,x_i)A_j=(SM_s)_{ij}A_j$ Neumann problrm $t(x_i)=\sum_j M^I(s_j,x_i)B_j=(SM_{sx})_{ij}B_j$

* NDIF method by Kang is the special case

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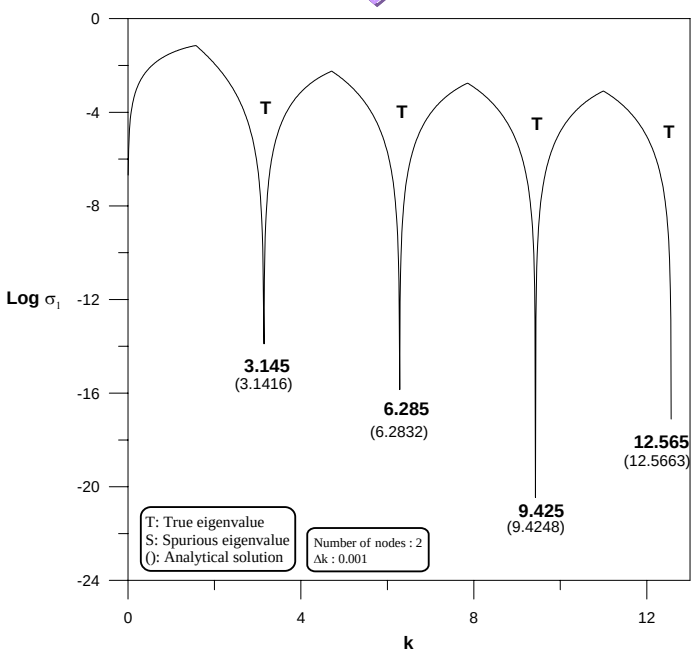
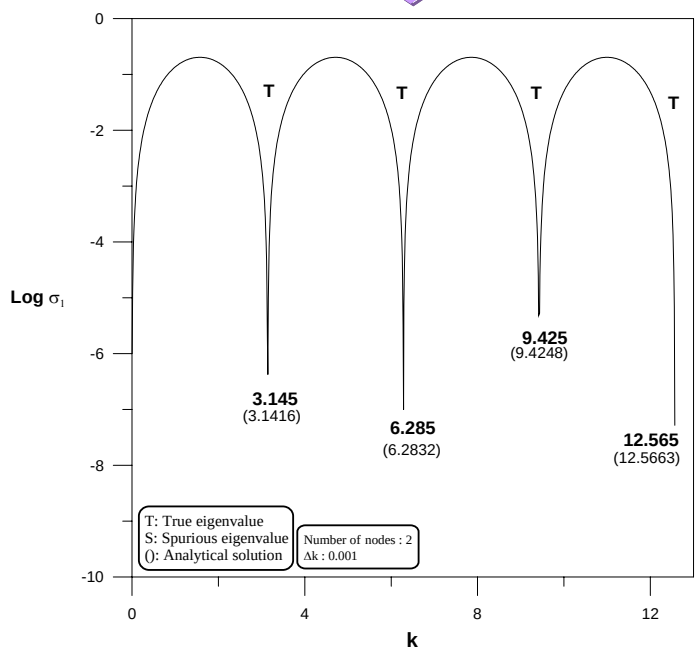
One-dimensional duct --- Dirichlet problem

Single-layer potential approach

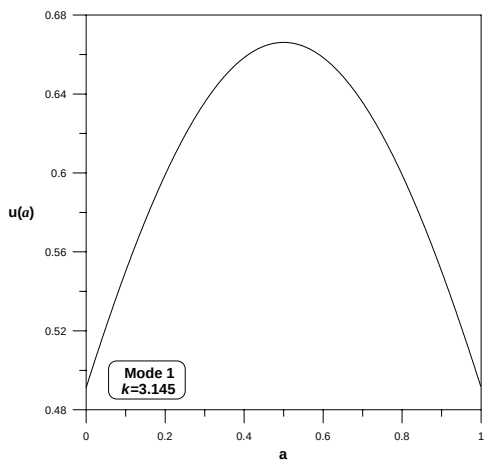


Double-layer potential approach

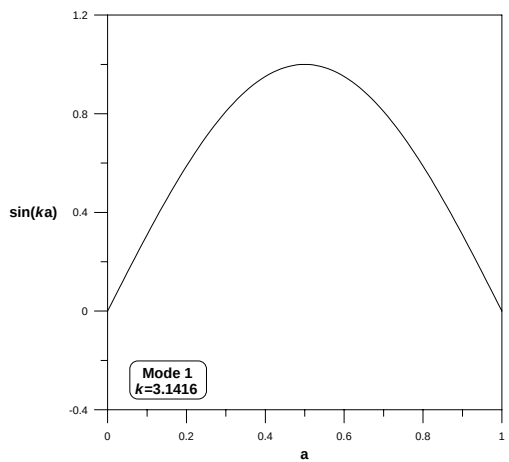


One-dimensional duct --- Dirichlet problem

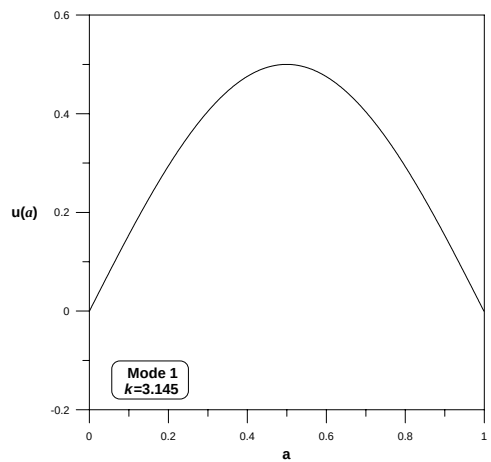
Single-layer potential approach



Double-layer potential approach

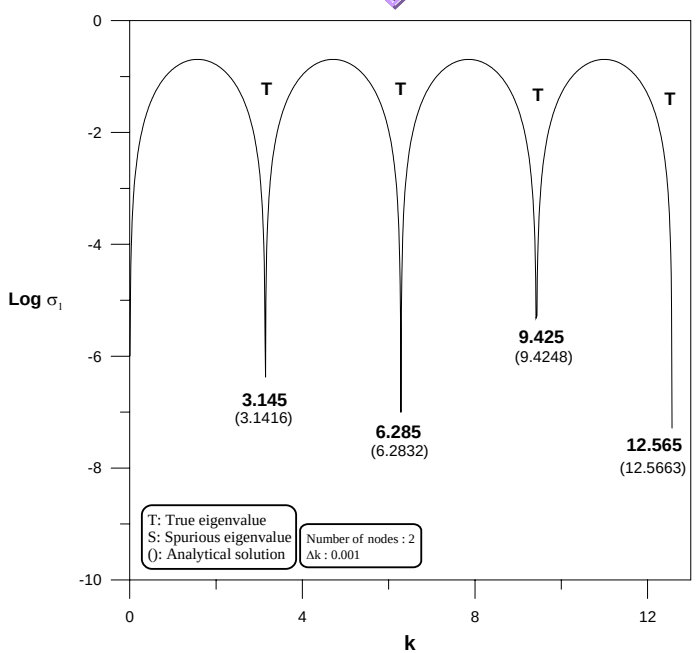


Analytical mode

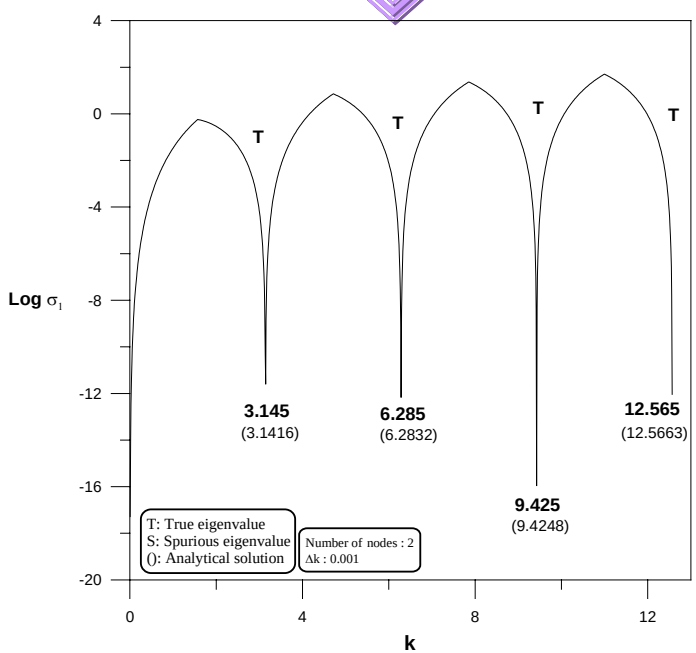


One-dimensional duct --- Neumann problem

Single-layer potential approach

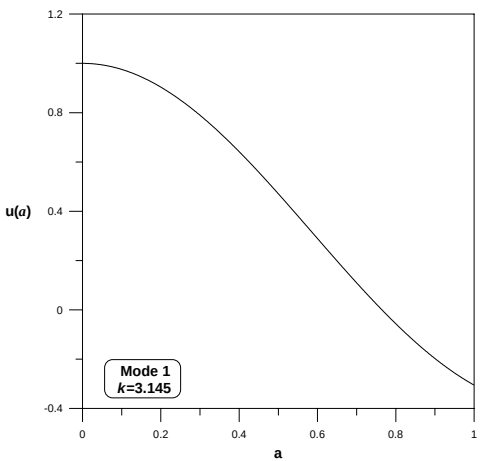


Double-layer potential approach

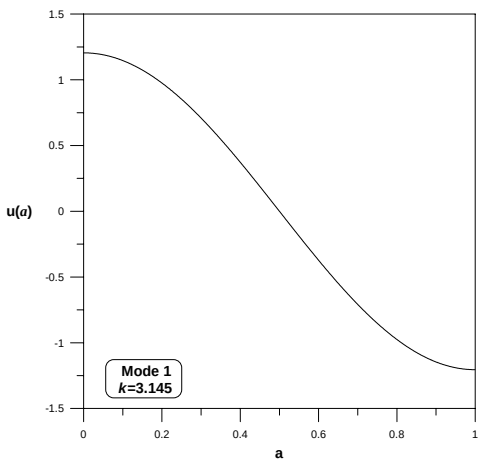
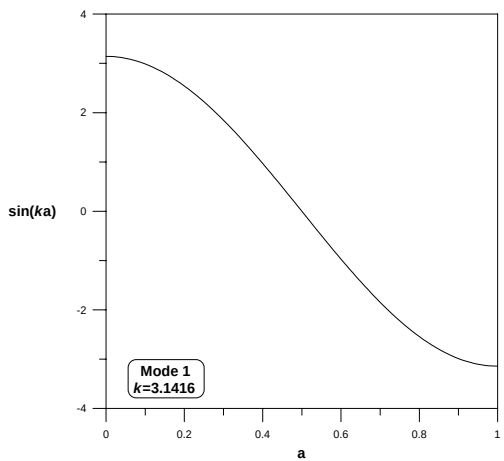


One-dimensional duct --- Neumann problem

Single-layer potential approach



Double-layer potential approach



Analytical mode