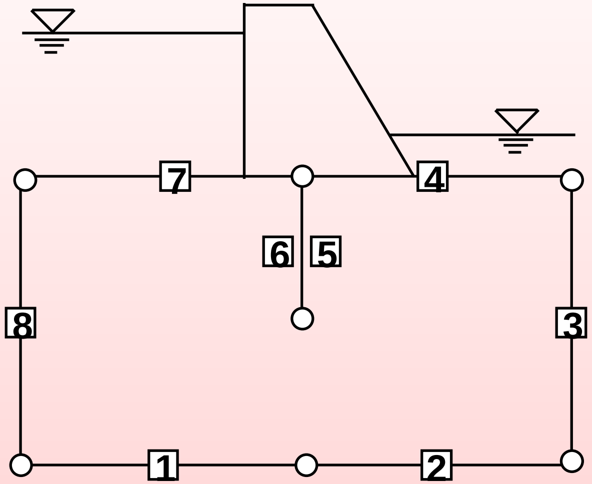
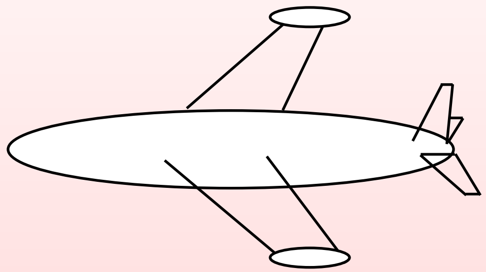
Hypersingular integral equation and its applications to computational mechanics

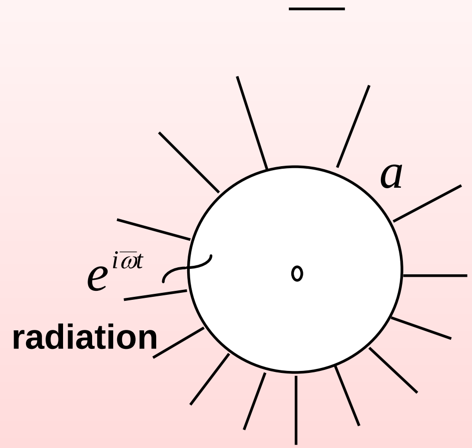
Seepage with sheetpiles



Thin-airfoill Aerodynamics



Radiation problem



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Presented by I. L. Chen

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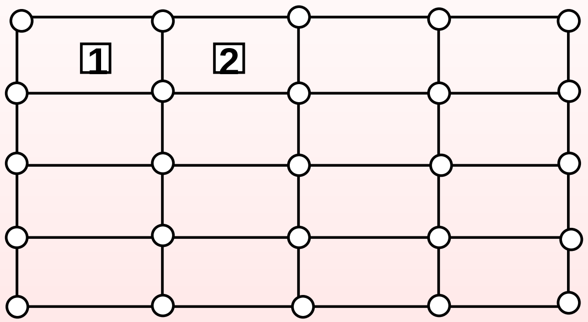
Feb.,19, 2002

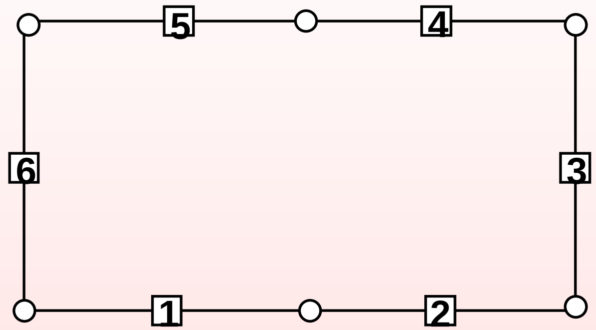
Outlines

- Introduction to BEM
- Introduction to dual BEM
- Theory of dual integral equations
- The role of dual integral equations
- Applications

  1. fictitious frequency (Helmholtz equation)
  2. spurious eigenvalue (Helmholtz equation)
  3. degenerate boundary (Laplace and Helmholtz)
  4. corner problem (Laplace and Helmholtz)
  5. adaptive BEM (Laplace and Helmholtz)
  6. degenerate scale(Laplace equation)
- Conclusions

What Is Boundary Element Method ?





○ **geometry node**

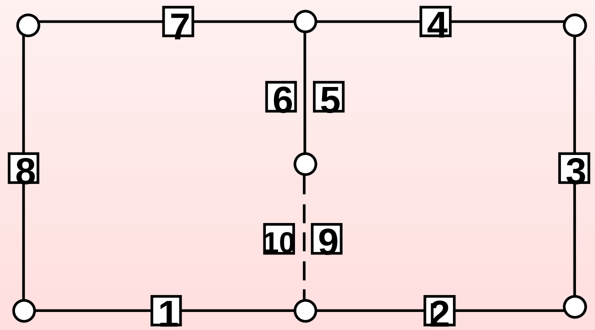
N **the Nth constant or linear element**

American doctor !

Chinese doctor !

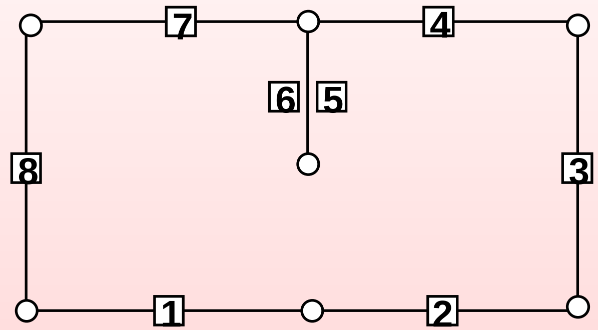
What Is Dual Boundary Element Method ?

- **Boundary element method**



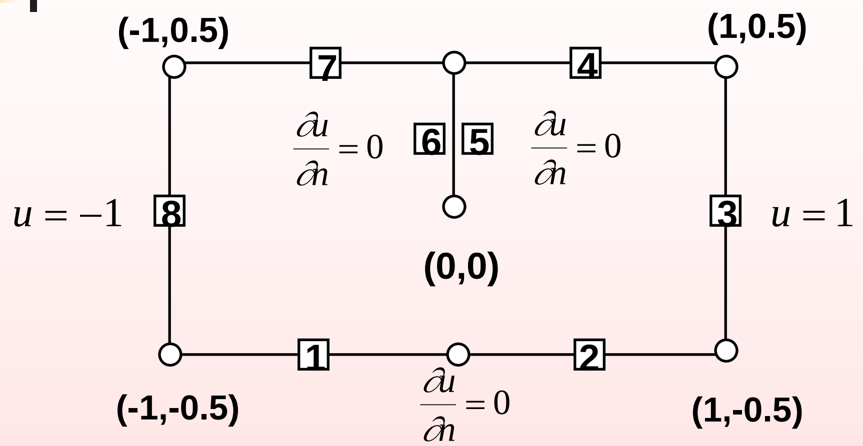
Artificial boundary introduced !

Dual boundary element method



Dual integral equations needed !

Degeneracy of the Degenerate Boundary



○ geometry node

N the Nth constant or linear element

$$[U]\{t\}=[T]\{u\}$$

$$[L]\{t\}=[M]\{u\}$$

5(+) 6(+)

$$[U]=\begin{bmatrix} -1.693 & -0.045 & 0.471 & 0.347 & -0.054 & -0.054 & 0.039 & -0.335 \\ -0.045 & -1.693 & -0.335 & 0.039 & -0.054 & -0.054 & 0.347 & 0.471 \\ 0.445 & -0.335 & -1.693 & -0.335 & 0.019 & 0.019 & 0.445 & 0.703 \\ 0.347 & 0.039 & -0.335 & -1.693 & -0.281 & -0.281 & -0.045 & 0.471 \\ -0.081 & -0.081 & 0.063 & -0.638 & -1.193 & -1.193 & -0.638 & 0.063 \\ -0.081 & -0.081 & 0.063 & -0.638 & -1.193 & -1.193 & -0.638 & 0.063 \\ 0.039 & 0.347 & 0.471 & -0.045 & -0.281 & -0.281 & -1.693 & -0.334 \\ -0.335 & 0.445 & 0.703 & 0.445 & 0.019 & 0.019 & -0.335 & -1.693 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ 5(+) \\ 6(+) \\ \\ \\ \end{matrix}$$

$n(s)$ 5(+) 6(-)

$$[T]=\begin{bmatrix} -\pi & 0.000 & 0.588 & 0.519 & -0.321 & 0.321 & 0.927 & 1.107 \\ 0.000 & -\pi & 1.107 & 0.927 & 0.321 & -0.321 & 0.519 & 0.588 \\ 0.219 & 1.107 & -\pi & 1.107 & 0.464 & -0.464 & 0.219 & 0.490 \\ 0.519 & 0.927 & 1.107 & -\pi & 0.785 & -0.785 & 0.000 & 0.588 \\ 0.927 & 0.927 & 0.888 & 1.326 & -\pi & -\pi & 1.326 & 0.888 \\ 0.927 & 0.927 & 0.888 & 1.326 & -\pi & -\pi & 1.326 & 0.888 \\ 0.927 & 0.519 & 0.588 & 0.000 & -0.7854 & 0.785 & -\pi & 1.107 \\ 1.107 & 0.219 & 0.490 & 0.219 & -0.464 & 0.464 & 1.107 & -\pi \end{bmatrix} \begin{matrix} \\ \\ \\ \\ 5(+) \\ 6(+) \\ \\ \\ \end{matrix}$$

$n(x)$ 5(+) 6(+)

$$[L]=\begin{bmatrix} \pi & 0.000 & 0.184 & 0.519 & 0.458 & 0.458 & 0.927 & 0.805 \\ 0.000 & \pi & 0.805 & 0.927 & 0.458 & 0.458 & 0.519 & 0.184 \\ 0.612 & 0.805 & \pi & 0.805 & 0.464 & 0.464 & 0.612 & 0.490 \\ 0.519 & 0.927 & 0.805 & \pi & 0.347 & 0.347 & 0.000 & 0.184 \\ -0.511 & 0.511 & 0.888 & 1.417 & \pi & -\pi & -1.417 & -0.888 \\ 0.511 & -0.511 & -0.888 & -1.417 & -\pi & \pi & 1.417 & 0.888 \\ 0.927 & 0.519 & 0.184 & 0.000 & 0.347 & 0.347 & \pi & 0.805 \\ 0.805 & 0.612 & 0.490 & 0.612 & 0.464 & 0.464 & 0.805 & \pi \end{bmatrix} \begin{matrix} \\ \\ \\ \\ 5(+) \\ 6(-) \\ \\ \\ \end{matrix}$$

$n(s)$ $n(x)$ 5(+) 6(-)

$$[M]=\begin{bmatrix} 4.000 & -1.333 & -0.205 & -0.061 & 0.600 & -0.600 & -0.800 & -1.600 \\ -1.333 & 4.000 & -1.600 & -0.800 & -0.600 & 0.600 & -0.061 & -0.205 \\ -0.282 & -1.600 & 4.000 & -1.600 & -0.400 & 0.400 & -0.282 & -0.236 \\ -0.061 & -0.800 & -1.600 & 4.000 & -1.000 & 1.000 & -1.333 & -0.205 \\ 0.853 & -0.853 & -0.715 & -3.765 & 8.000 & -8.000 & 3.765 & 0.715 \\ -0.853 & 0.853 & 0.715 & 3.765 & -8.000 & 8.000 & -3.765 & -0.715 \\ -0.800 & -0.062 & -0.205 & -1.333 & 1.000 & -1.000 & 4.000 & -1.600 \\ -1.600 & -0.282 & -0.235 & -0.282 & 0.400 & -0.400 & -1.600 & 4.000 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ 5(+) \\ 6(-) \\ \\ \\ \end{matrix}$$

Theory of dual integral equations

$$f(x) = (x-a)^2 Q(x) + px + q$$

$$f(a) = pa + q, \quad when \; x = a$$

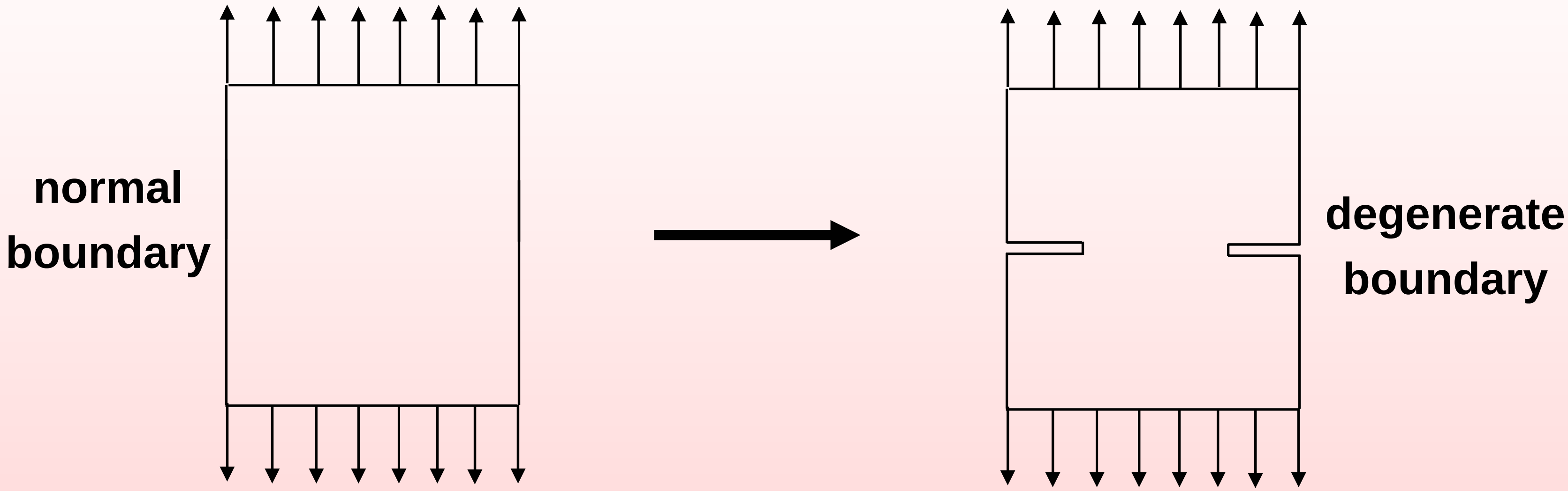
The constraint equation is not enough to determine the coefficient p and q,

Another constraint equation is required

$$f'(x) = 2(x-a)Q(x) + (x-a)^2 Q'(x) + p$$

$$f'(a) = p, \quad when \quad x = a$$

Dual Integral Equations by Hong and Chen(1984-1986)



Singular integral equation → Hypersingular integral equation

Cauchy principal value → Hadamard principal value

Boundary element method → Dual boundary element method

Dual integral equations

Singular integral equation

$$2\pi u(x) = \int_B T(s,x)u(s)\,dB(s) - \int_B U(s,x)t(s)\,dB(s), x \in D$$

Hypersingular integral equation

$$2\pi t(x) = \int_B M(s,x)u(s)\,dB(s) - \int_B L(s,x)t(s)\,dB(s), x \in D$$

where $U(s,x)$ **is the fundamental solution .**

$$T(s,x) \equiv \frac{\partial U}{\partial s} \qquad L(s,x) \equiv \frac{\partial U}{\partial x} \qquad M(s,x) \equiv \frac{\partial^2 U}{\partial s \partial x}$$

Roles of hypersingularity in boundary element method

complementary constraints

fictitious frequency
1. exterior radiation
2. scattering

spurious eigenvalue
1. Real-part BEM
2. Imaginary-part BEM
3. MRM

degenerate boundary
1. cutoff wall
2. sheet pile
3. crack
4. baffle
5. thin airfoil
6. antenna

(o) $\begin{cases}[U]\{t\}=[T]\{u\}\\ [L^-]\{t\}=[M^-]\{u\}\end{cases}$

(o) $\begin{cases}[U]\{t\}=[T]\{u\}\\ [L^+]\{t\}=[M^+]\{u\}\end{cases}$

(x) $\begin{cases}[L^-]\{t\}=[M^-]\{u\}\\ [L^+]\{t\}=[M^+]\{u\}\end{cases}$

corner problem
1. $\begin{cases}[U]\{t\}=[T]\{u\}\\ [L^-]\{t\}=[M^-]\{u\}\end{cases}$
2. $\begin{cases}[U]\{t\}=[T]\{u\}\\ [L^+]\{t\}=[M^+]\{u\}\end{cases}$
3. $\begin{cases}[L^-]\{t\}=[M^-]\{u\}\\ [L^+]\{t\}=[M^+]\{u\}\end{cases}$

adaptive BEM
1. error estimator

degenerate scale

Helmholtz equation

$$(\nabla^2 + k^2)u(x) = 0, \quad x \in D$$

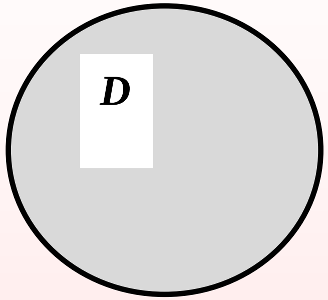
u **: acoustic potential**

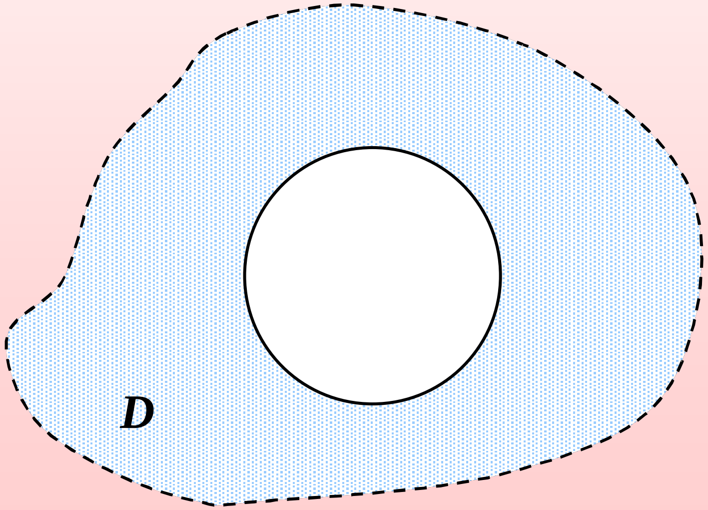
k **: wave number,** $k = \omega / c$

ω **: angular frequency**

c **: sound speed**

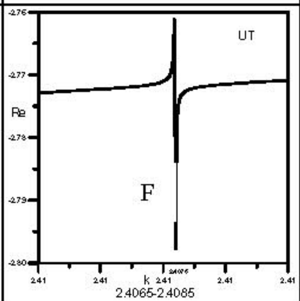
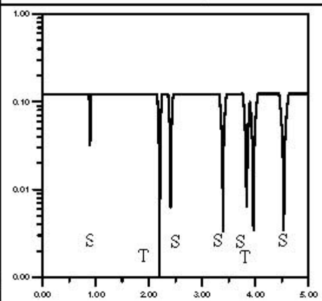
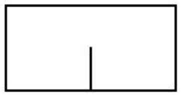
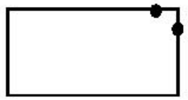
D **: domain of interest**





Four pitfalls in rank-deficiency problems using BEM

Table 1-1 The rank-deficiency problems using BEM.

Physical problems	Exterior acoustics	Interior acoustics (Eigen problem)	Degenerate boundary	Corner problem
Mathematical formulation & numerical method	Complex-valued BEM (UT or LM)	BEM (real-part BEM or imaginary-part BEM MRM)	Complex-valued BEM (UT or LM)	BEM (UT or LM)
Numerical trouble			 $[T]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{17} \\ \times & \times & \times & \cdots & \times \\ \times & \times & \times & \cdots & \times \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{71} & a_{72} & a_{73} & \cdots & a_{77} \end{bmatrix}$	 $[U]=\begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{17} \\ \times & \times & \times & \cdots & \times \\ \times & \times & \times & \cdots & \times \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{71} & b_{72} & b_{73} & \cdots & b_{77} \end{bmatrix}$
Treatment	1. Dual BEM 2. CHIEF method 3. Burton and Miller method	1. Dual BEM 2. CHEEF method 3. SVD updating technique	1. Dual BEM	1. Dual BEM

Mathematical tools

1. **Degenerate kernels**
2. **Circulants**
3. **Fredholm alternative theorem**
4. **SVD updating term and updating document**

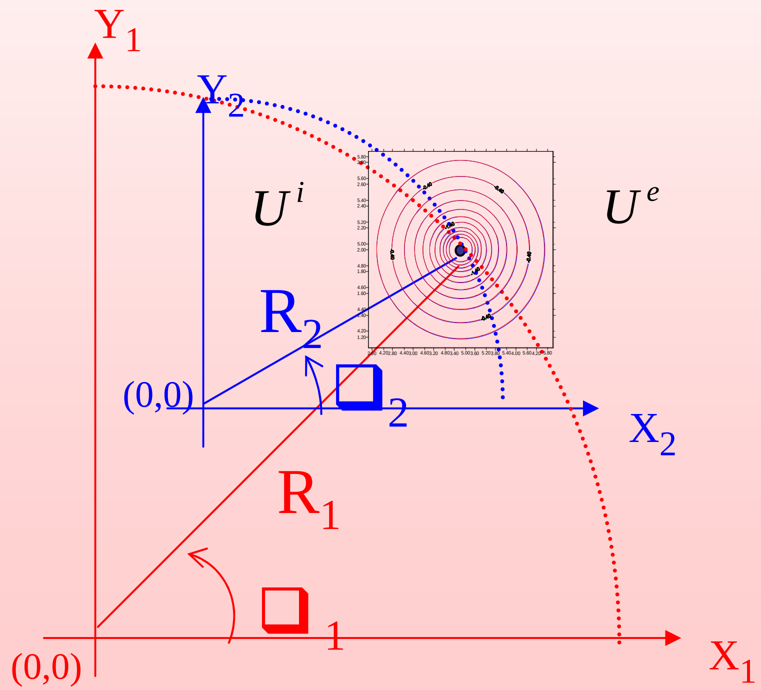
Degenerate kernels

$$U(s,x)=\frac{-i\pi H_0^{(1)}(kr)}{2} \quad \textbf{(closed-form)}$$

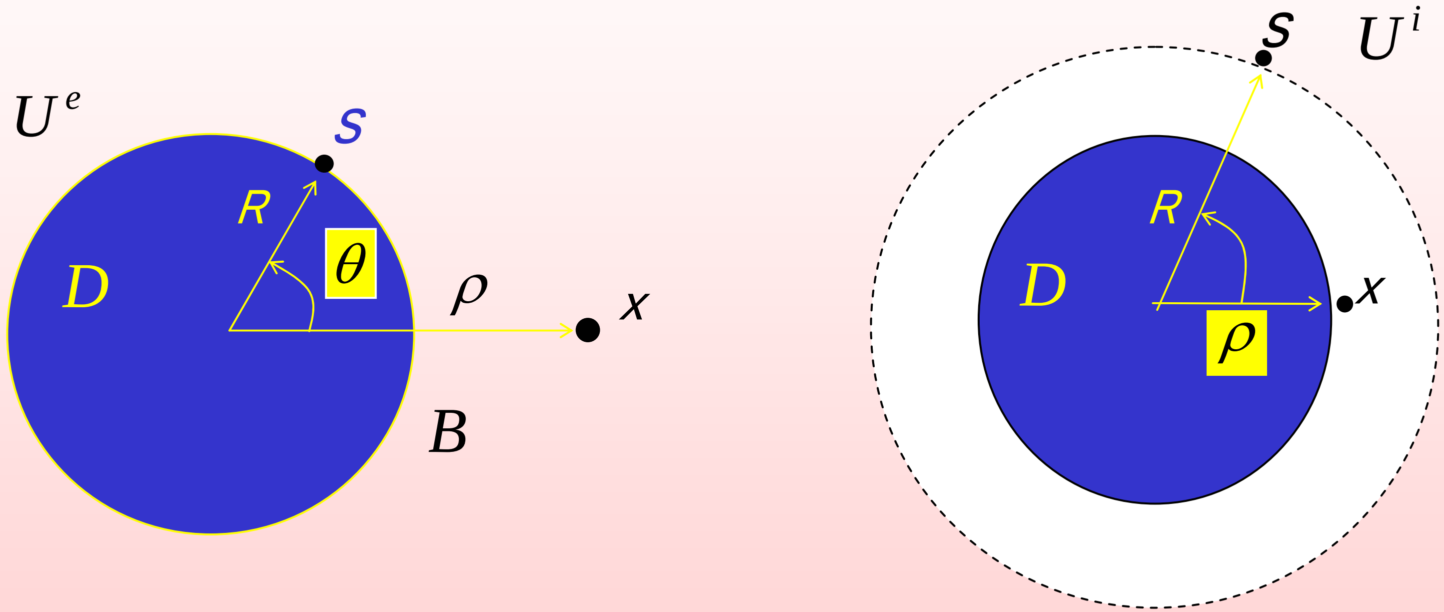
$$U(s,x)=\begin{cases} U^i(R,\theta;\rho,\phi)=\sum_{n=-\infty}^{\infty}\frac{\pi}{2}[-iJ_n(kR)+Y_n(kR)]J_n(k\rho)\cos(n(\theta-\phi)),\ R>\rho \\ U^e(R,\theta;\rho,\phi)=\sum_{n=-\infty}^{\infty}\frac{\pi}{2}[-iJ_n(k\rho)+Y_n(k\rho)]J_n(kR)\cos(n(\theta-\phi)),\ R<\rho \end{cases}$$

$s=(R,\theta)$ **: source point**

$x=(\rho,\phi)$ **: field point**



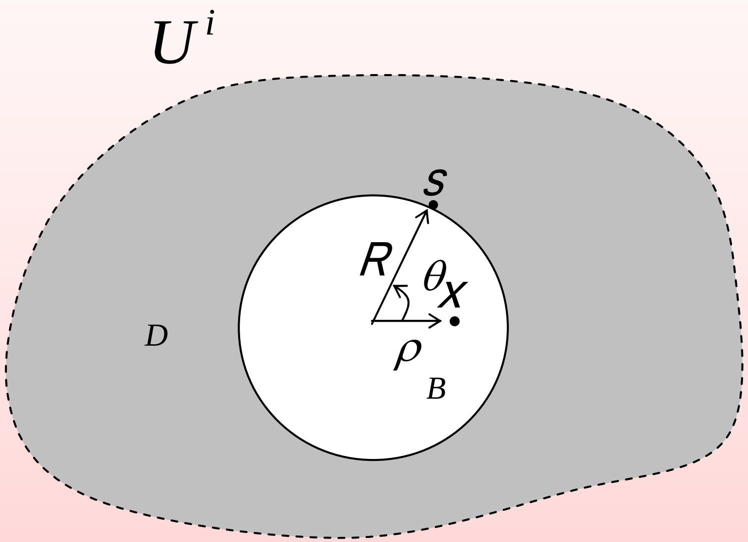
Choice of U^i and U^e for interior problems

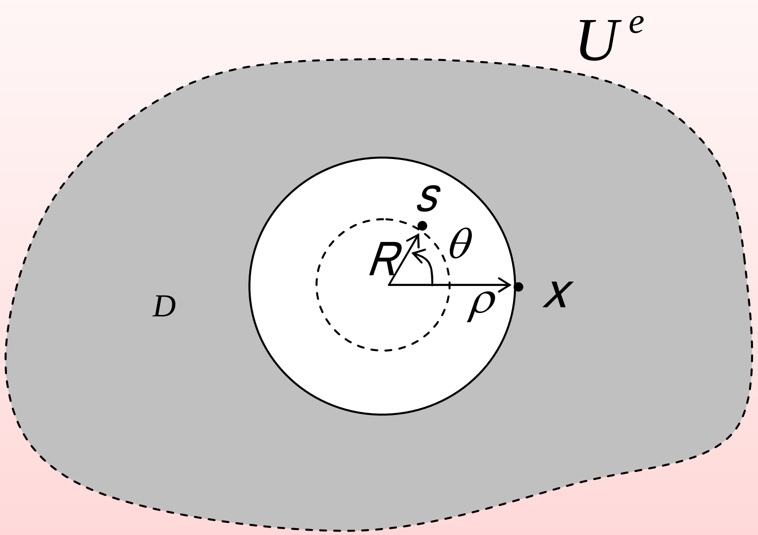


Direct method

Indirect method

Choice of U^i or U^e for exterior problems



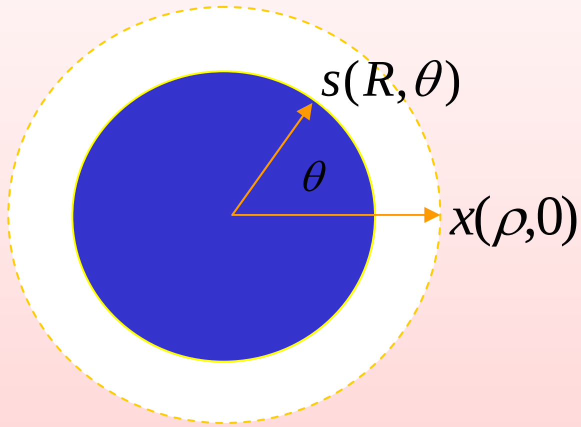


Direct method **Indirect method**

Circulants

Discretizing 2N constant elements for a circular boundary

$$[G]=\begin{bmatrix} a_0 & a_1 & \cdots & a_{2N-1} \\ a_{2N-1} & a_0 & \cdots & a_{2N-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix}$$



G **can be** $U^i, U^e, T^i, T^e, L^i, L^e, M^i$ **or** M^e

$$T(s,x)\equiv\frac{\partial U}{\partial n_s} \qquad L(s,x)\equiv\frac{\partial U}{\partial n_x} \qquad M(s,x)\equiv\frac{\partial^2 U}{\partial n_s \partial n_x}$$

Properties of circulants

$$a_m = \int_{(m-1/2)\Delta\theta}^{(m+1/2)\Delta\theta} G(\theta,0)Rd\theta \approx G(\theta_m,0)R\Delta\theta, \quad m = 0,1,2\cdots,2N-1$$

$$\Delta\theta = \frac{2\pi}{2N}, \quad \theta_m = m\Delta\theta$$

$[\mathrm{G}]$ **is the circulants.**

$$[G] = a_0 I + a_1 C_{2N}^1 + a_2 C_{2N}^2 + \cdots + a_{2N-1} C_{2N}^{2N-1}$$

$$C_{2N} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{2N\times 2N}$$

Eigenvalue and eigenvector of circulants

The **eigenvalue** for the circulants is the root for $\alpha^{2N} = 1$

$$\alpha_n = e^{i\frac{2\pi n}{2N}}, \quad n = 0, \pm 1, \pm 2, \cdots, \pm (N-1), N$$

The **eigenvector** is $\{\phi_n\} = \begin{Bmatrix} \alpha_n^0 \\ \alpha_n^1 \\ \alpha_n^2 \\ \vdots \\ \alpha_n^{2N-1} \end{Bmatrix}$

Determinants of the influence matrices

$$\det[U^i] = \lambda_0 \lambda_N (\lambda_1 \cdots \lambda_{N-1})(\lambda_{-1} \cdots \lambda_{-(N-1)})$$

$$\det[U^e] = \lambda_0 \lambda_N (\lambda_1 \cdots \lambda_{N-1})(\lambda_{-1} \cdots \lambda_{-(N-1)})$$

$$\det[T^e] = \mu_0 \mu_N (\lambda\mu \cdots \mu_{N-1})(\mu_{-1} \cdots \mu_{-(N-1)})$$

$$\det[L^i] = \mu_0 \mu_N (\lambda\mu \cdots \mu_{N-1})(\mu_{-1} \cdots \mu_{-(N-1)})$$

$$\det[T^i] = \upsilon_0 \upsilon_N (\upsilon_1 \cdots \upsilon_{N-1})(\upsilon_{-1} \cdots \upsilon_{-(N-1)})$$

$$\det[L^e] = \upsilon_0 \upsilon_N (\upsilon_1 \cdots \upsilon_{N-1})(\upsilon_{-1} \cdots \upsilon_{-(N-1)})$$

$$\det[M^i] = k_0 k_N (k_1 \cdots k_{N-1})(k_{-1} \cdots k_{-(N-1)})$$

$$\det[M^e] = k_0 k_N (k_1 \cdots k_{N-1})(k_{-1} \cdots k_{-(N-1)})$$

$$l = 0, \pm 1, \pm 2, \cdots, \pm (N-1), N$$

Eigenvalue of the influence matrices

$$\lambda_l = \pi^2 \rho(-iJ_l(k\rho) + Y_l(k\rho))J_l(k\rho),$$

$$\mu_l = \pi^2 k\rho(-iJ_l(k\rho) + Y_l(k\rho))J_l'(k\rho),$$

$$\upsilon_l = \pi^2 k\rho(-iJ_l'(k\rho) + Y_l'(k\rho))J_l(k\rho),$$

$$k_l = \pi^2 k^2 \rho(-iJ_l'(k\rho) + Y_l'(k\rho))J_l'(k\rho).$$

$$l = 0, \pm 1, \pm 2, \cdots \pm (N-1), N$$

Fredholm alternative theorem

For solving an algebraic system:

$[A]\{x\} = \{b\} \neq \{0\}$ → $\{x\}$ **has a unique solution,**

If and only if

→ $[A]\{x\} = \{0\}$ ⇒ $\{x\} = \{0\}$

Alternatively

if

→ $\{x\}$ **has at least one solution,**

→ $[A]^{+}\{\phi\} = \{0\}$ ⇒ $\{\phi\}$ **: Nontrivial solution**

⇓

$\{b\}^{+}\{\phi\} = 0$

"+" denotes transpose conjugate

Singular value decomposition (SVD)

$$[U]_{m\times n} = \Phi_{m\times m}\Sigma_{m\times n}\Psi^{+}_{n\times n}$$

Ψ : right unitary matrix

Φ : left unitary matrix

$$[\Sigma] = \begin{bmatrix} \sigma_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}_{m\times n} \quad , \quad m > n$$

$$\sigma_n > \sigma_{n-1} > \cdots > \sigma_1$$

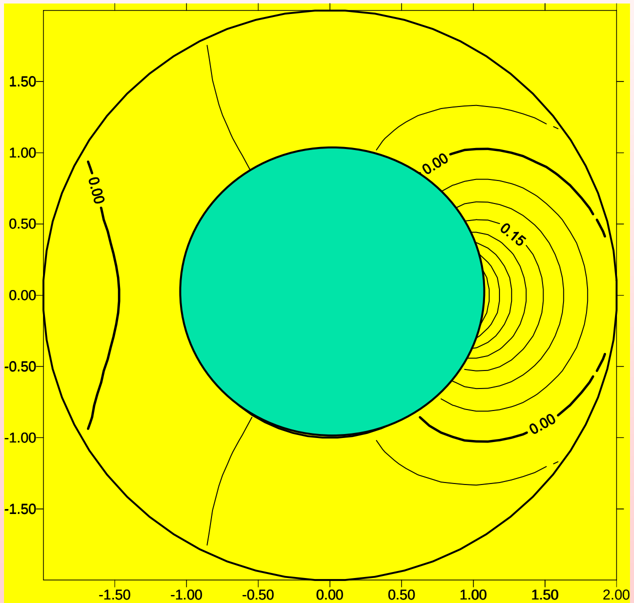
SVD updating term and updating document

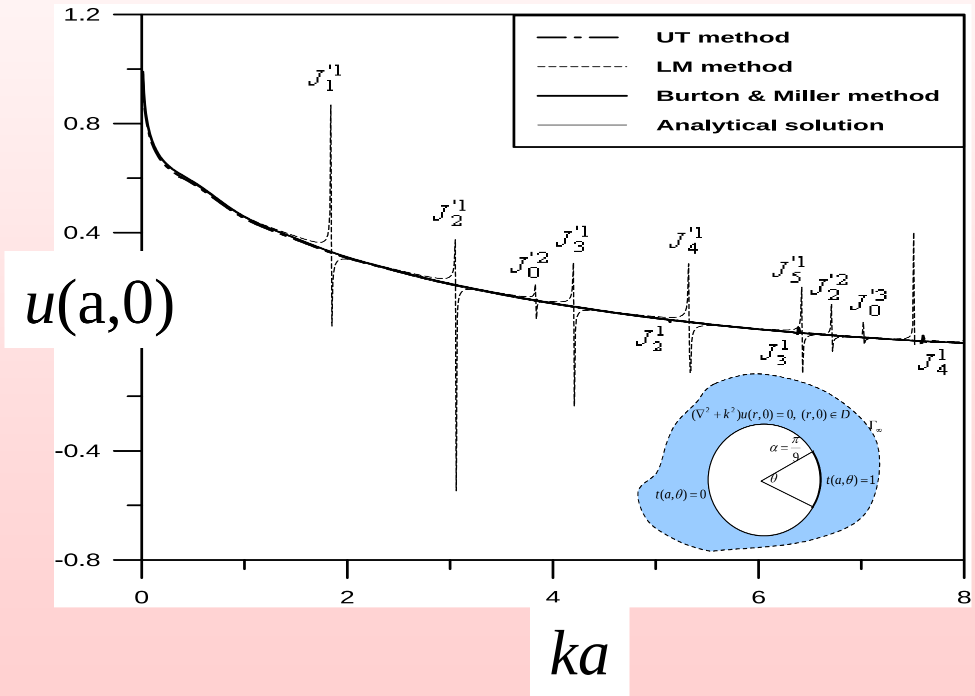
Dirichlet B.C.	Neumann B.C.
updating term $$\begin{bmatrix} U_R \\ L_R \end{bmatrix} \{\varphi_D\} = \{0\}$$	**updating term** $$\begin{bmatrix} T_R \\ M_R \end{bmatrix} \{\varphi_N\} = \{0\}$$
updating document $$\{\varphi_D^T\}\begin{bmatrix} U_R^T & L_R^T \end{bmatrix} = \{0\}$$	**updating document** $$\{\varphi_N^T\}\begin{bmatrix} T_R^T & M_R^T \end{bmatrix} = \{0\}$$

Four pitfalls in rank-deficiency problems using BEM

1. Fictitious frequency
2. Spurious eigenvalue
3. Degenerate boundary
4. Corner problem

Fictitious frequency (exterior problem)





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Radiation problem with Neumann B. C., $t=\bar{t}$

1.Singular equation (*UT*)

$$[T^i]\{u\}=[U^i]\{\bar{t}\}=\{p_1\}$$

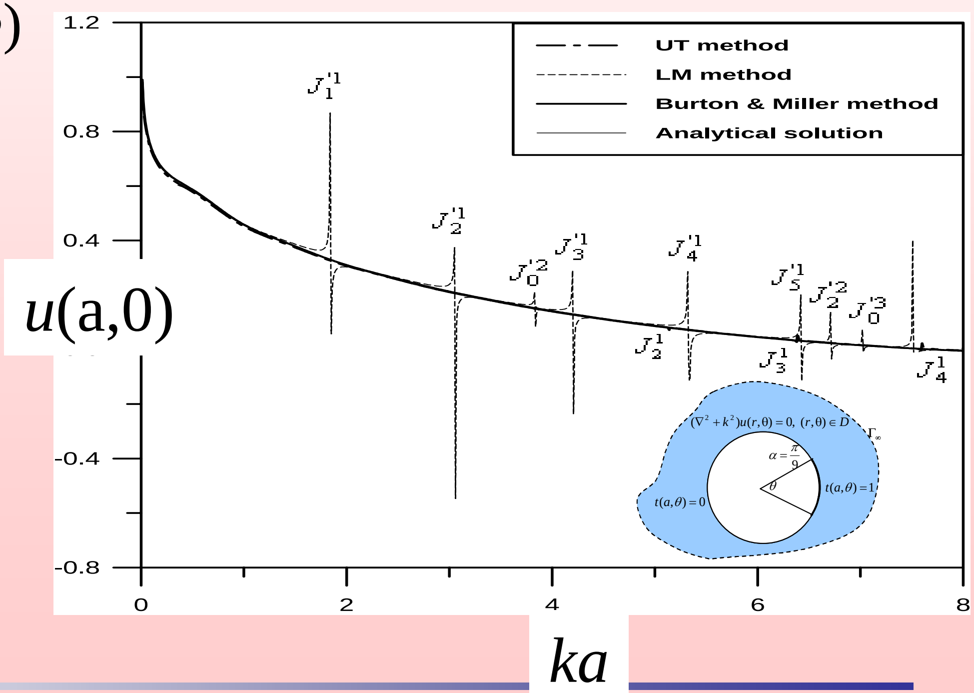
$$\{u\}=[T^i]^{-1}\{p_1\}$$

$$\upsilon_l=\pi^2 k\rho(-iJ_l^{'}(k\rho)+Y_l^{'}(k\rho))J_l(k\rho)$$

$$(-iJ_l^{'}(k\rho)+Y_l^{'}(k\rho))J_l(k\rho)=0$$

$$\because -iJ_l^{'}(k\rho)+Y_l^{'}(k\rho)\neq 0$$

$$J_l(k\rho)=0$$



Radiation problem with Neumann B. C.,

2.Hypersingular equation (*LM*)

$$[M^i]\{u\} = [L^i]\{\bar{t}\} = \{p_2\},$$

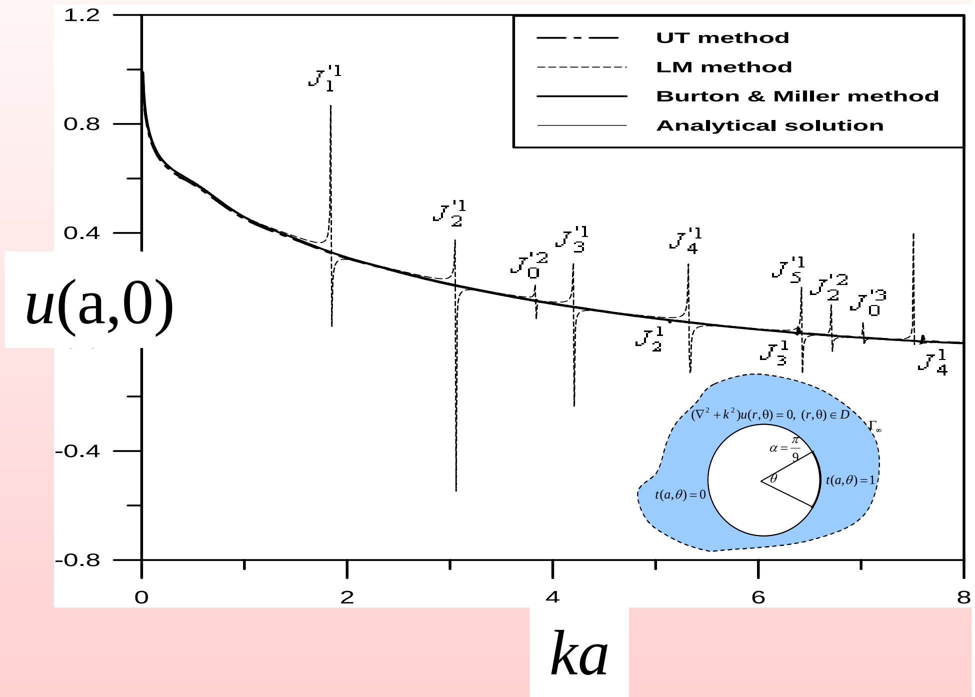
$$\{u\} = [M^i]^{-1}\{p_2\},$$

$$k_l = \pi^2 k^2 \rho(-iJ_l^{'}(k\rho) + Y_l^{'}(k\rho))J_l'(k\rho)$$

$$(-iJ_l^{'}(k\rho) + Y_l^{'}(k\rho))J_l^{'}(k\rho) = 0$$

$$\because -iJ_l^{'}(k\rho) + Y_l^{'}(k\rho) \neq 0$$

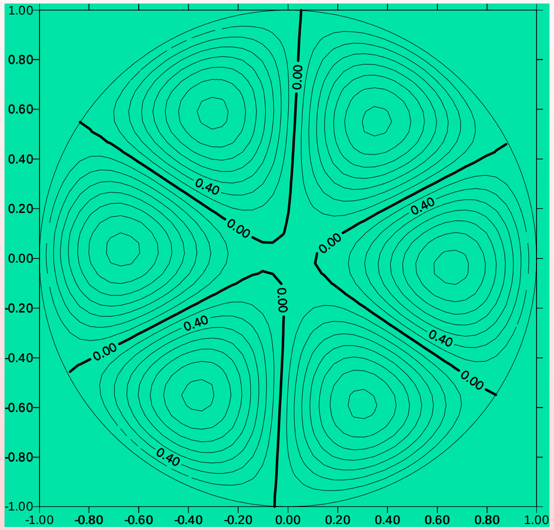
$$J_l^{'}(k\rho) = 0$$



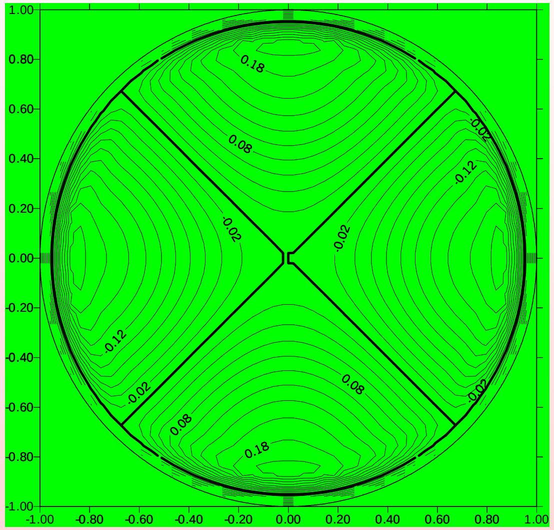
Occurrence of fictitious frequency

	Direct method		**Indirect method**	
	UT	*LM*	*UL*	*TM*
Dirichlet B.C.	$J_n(k\rho)$	$J'_n(k\rho)$	$J_n(k\rho)$	$J'_n(k\rho)$
Neumann B.C.	$J_n(k\rho)$	$J'_n(k\rho)$	$J_n(k\rho)$	$J'_n(k\rho)$

Spurious eigenvalue (interior problem)



True mode



Spurious mode

Interior problem with Dirichlet B.C. $u = 0$

a. **Complex-valued** **BEM** ***UT*** **equation**

$$[U^e]\{t\} = [T^e]\{u\} = 0,$$

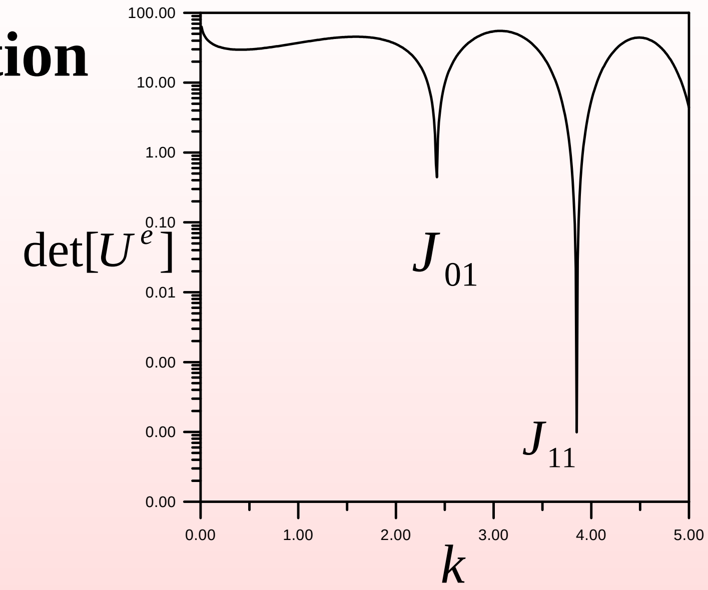
The eigenequation is derived

$$\therefore (-iJ_l(k\rho) + Y_l(k\rho))J_l(k\rho) = 0$$

$$-iJ_l(k\rho) + Y_l(k\rho) \neq 0$$

The true eigenvalues are the roots of

$$J_l(k\rho) = 0$$



Interior problem with Dirichlet B.C. $u = 0$

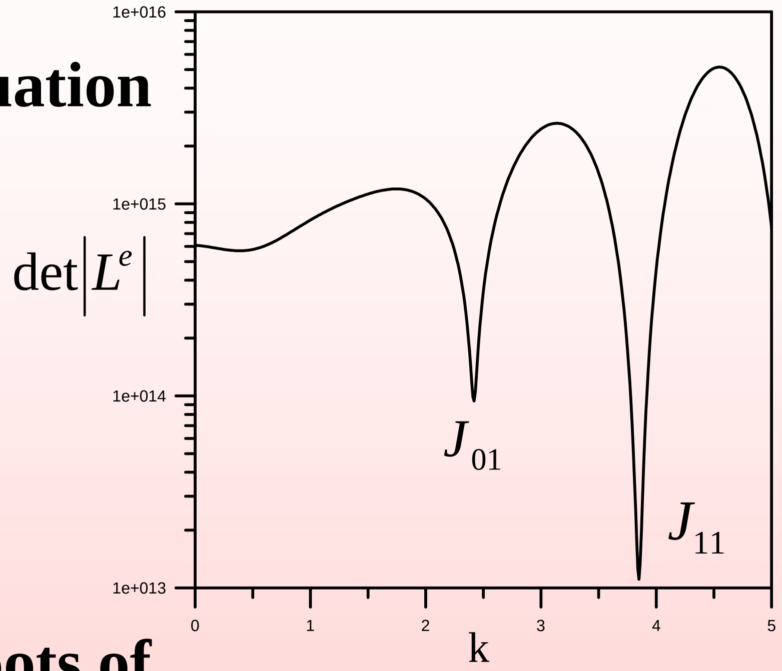
b. Complex-valued BEM *LM* equation

$$[L^e]\{t\} = [M^e]\{u\} = 0,$$

The eigenequation is derived

$$\therefore (-iJ'_l(k\rho) + Y'_l(k\rho))J_l(k\rho) = 0$$

$$-iJ'_l(k\rho) + Y'_l(k\rho) \neq 0$$



The true eigenvalues are the roots of

$$J_l(k\rho) = 0$$

Interior problem with Dirichlet B.C. $u = 0$

Real-part BEM

a. The *UT* equation

$$[U_R^e]\{t\} = [T_R^e]\{u\} = 0$$

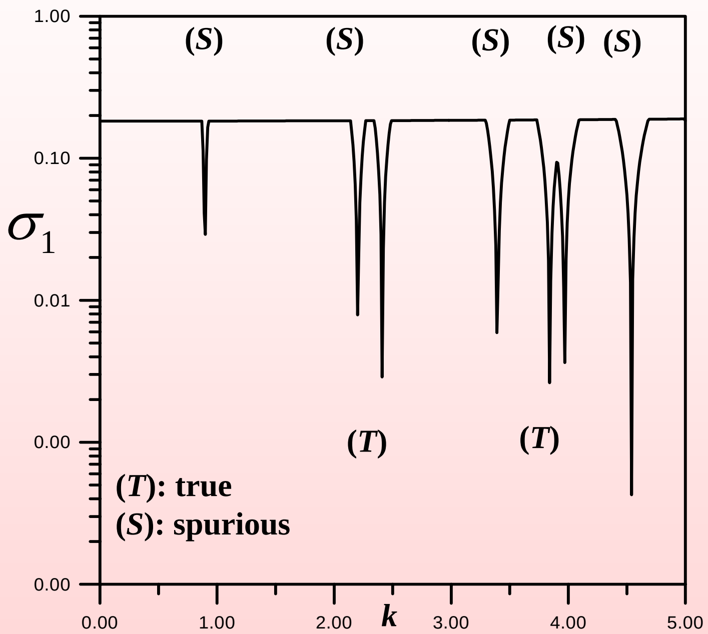
***R* denotes the real part.**

The true and spurious eigenvalues are the roots of

$$Y_l(k\rho)J_l(k\rho) = 0$$

$J_l(k\rho) = 0$, **true eigenvalue**

$Y_l(k\rho) = 0$, **spurious eigenvalue**

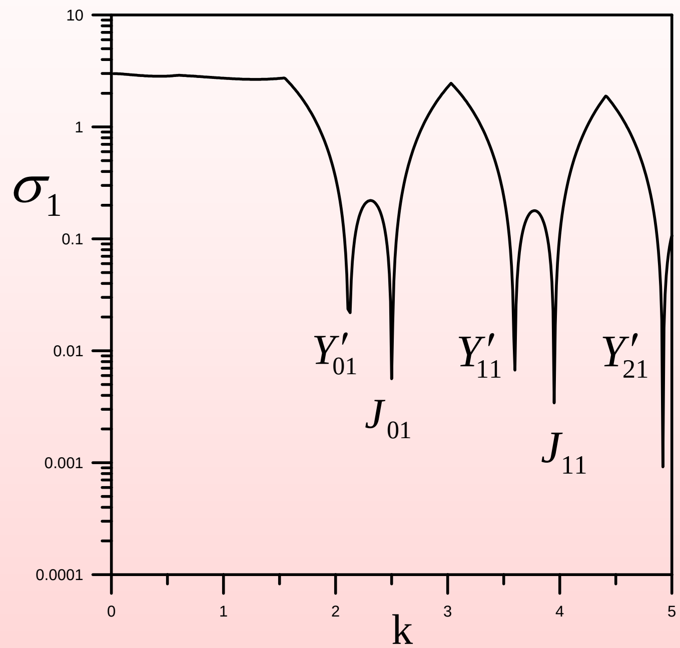


Interior problem with Dirichlet B.C. $u=0$

Real-part BEM

b. The *LM* equation

$$[L_R^e]\{t\}=[M_R^e]\{u\}=0$$



The true and spurious eigenvalues are the roots of

$$Y'_l(k\rho)J_l(k\rho)=0$$

$J_l(k\rho)=0$, **true eigenvalue**

$Y'_l(k\rho)=0$, **spurious eigenvalue**

		Direct method					
		UT formulation			*LM* formulation		
		Comp. valued BEM	Real-part BEM	Imag.-part BEM	Comp. valued BEM	Real-part BEM	Imag.-part BEM
Dirichlet B.C.	True	J_n	J_n	J_n	J_n	J_n	J_n
	Spurious		Y_n	J_n *		Y'_n	J'_n
Neumann B.C.	True	J'_n	J'_n	J'_n	J'_n	J'_n	J'_n
	Spurious		Y_n	J_n		Y'_n	J'_n *

* Denote the spurious multiplicity

Filter out the spurious eigenvalue

UT **formulation :** $[U_R^e]\{t\}=[T_R^e]\{u\}=\{b\}$

k_s **: spurious eigenvalue**

Neumann problem $(t=\bar{t})$

$\Rightarrow [T_R^e]\{u\}=\{b\}$

$\{b^T\}\{\phi_s\}=0$

$[T^T]\{\phi_s\}=\{0\}$

$\{b^T\}\{\phi_s\}=\{\bar{t}^T\}[U_R^e]^T\{\phi_s\}=0$

$$\Rightarrow \begin{bmatrix} U^T \\ T^T \end{bmatrix}\{\phi_s\}=\{0\}$$

Fredholm Alternative therrem

Dirichlet problem $(u=\bar{u})$

$\Rightarrow [U_R^e]\{t\}=\{b\}$

$\{b^T\}\{\phi_s\}=0$

$[U^T]\{\phi_s\}=\{0\}$

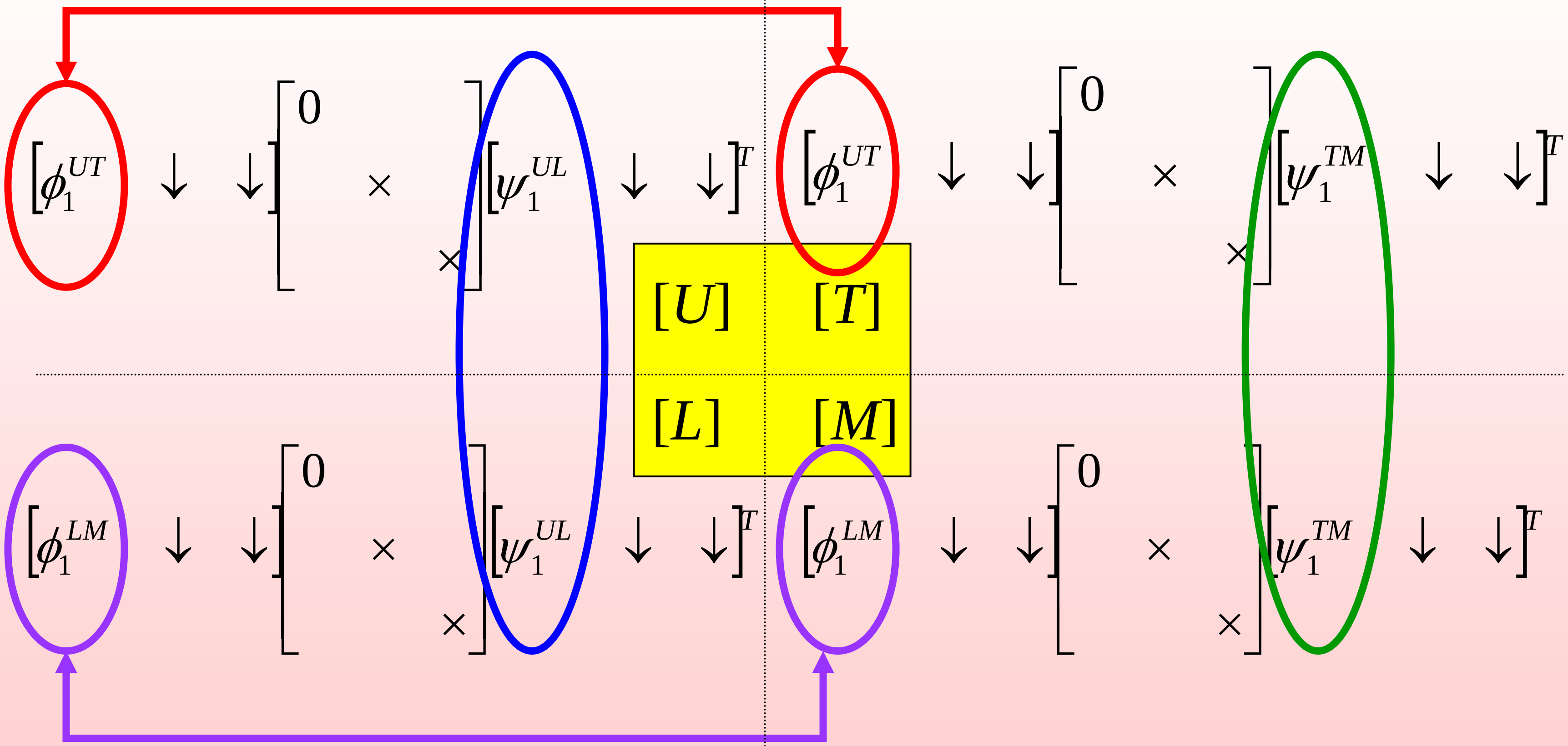
$\{b^T\}\{\phi_s\}=\{\bar{u}^T\}\{T^T\}\{\phi_s\}=0$

$$\Rightarrow \begin{bmatrix} U^T \\ T^T \end{bmatrix}\{\phi_s\}=\{0\}$$

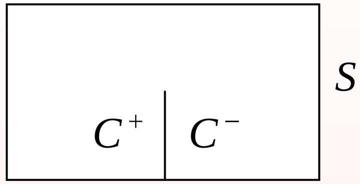
$\{\phi_s\}$ **: spurious mode**

U^T **is transpose matrix of** U_R^e

SVD structure in *U*, *T*, *L* and *M* matrices



Degenerate boundary



$$[K_U]\{t\} = \begin{bmatrix} U_{iSjS} & U_{iSjC^+} & U_{iSjC^-} \\ U_{iC^+jS} & U_{iC^+jC^+} & U_{iC^+jC^-} \\ U_{iC^-jS} & U_{iC^-jC^+} & U_{iC^-jC^-} \end{bmatrix}\{t\}$$

$$[K_L]\{t\} = \begin{bmatrix} L_{iSjS} & L_{iSjC^+} & L_{iSjC^-} \\ L_{iC^+jS} & L_{iC^+jC^+} & L_{iC^+jC^-} \\ L_{iC^-jS} & L_{iC^-jC^+} & L_{iC^-jC^-} \end{bmatrix}\{t\}$$

$$[K] = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{17} \\ \times & \times & \times & \cdots & \times \\ \times & \times & \times & \cdots & \times \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{71} & b_{72} & b_{73} & \cdots & b_{77} \end{bmatrix}$$

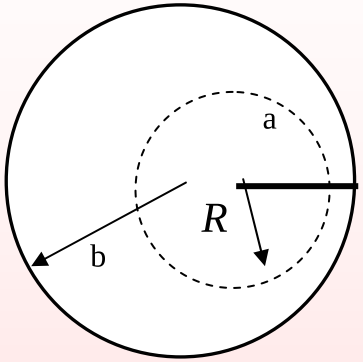
Adding constraints

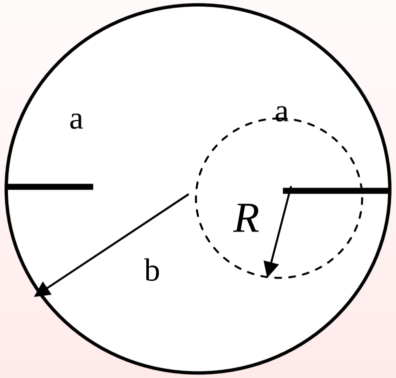
Adding constraints

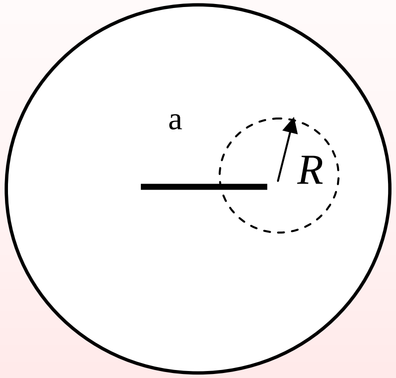
$$[K_{UL}]\{t\} = \begin{bmatrix} U_{iSjS} & U_{iSjC^+} & U_{iSjC^-} \\ U_{iC^+jS} & U_{iC^+jC^+} & U_{iC^+jC^-} \\ L_{iC^-jS} & L_{iC^-jC^+} & L_{iC^-jC^-} \end{bmatrix}\{t\}$$

$$[K_{LU}]\{t\} = \begin{bmatrix} L_{iSjS} & L_{iSjC^+} & L_{iSjC^-} \\ L_{iC^+jS} & L_{iC^+jC^+} & L_{iC^+jC^-} \\ U_{iC^-jS} & U_{iC^-jC^+} & U_{iC^-jC^-} \end{bmatrix}\{t\}$$

Dynamic stress intensity factor



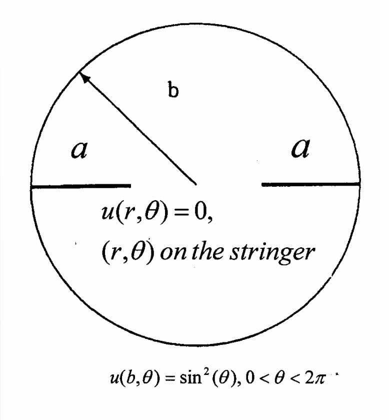


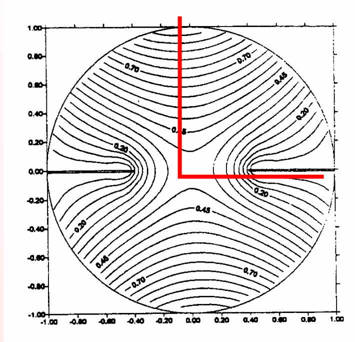


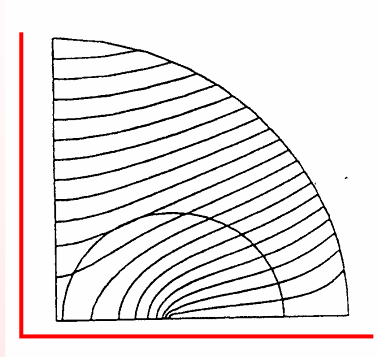
$$K_w = \frac{k\sqrt{R/2\pi}}{\sin(kR)} \int_0^{2\pi} \sin(\frac{\theta'}{2})\phi(R,\theta')d\theta'$$

R :the radius enclosing the singularity

Numerical examples

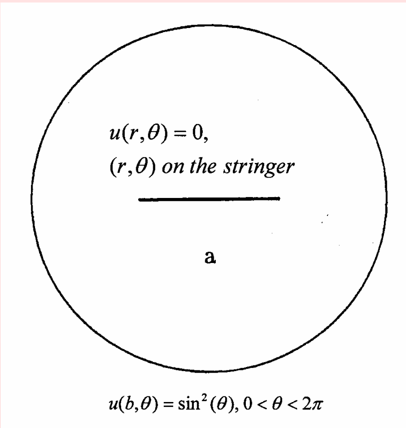


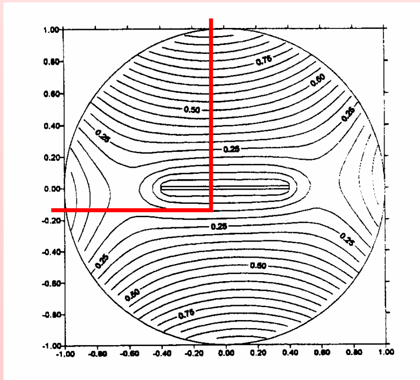


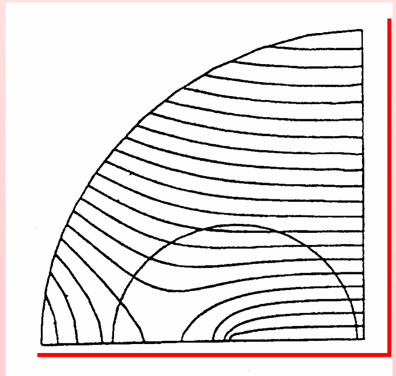


Dual BEM

DtN method





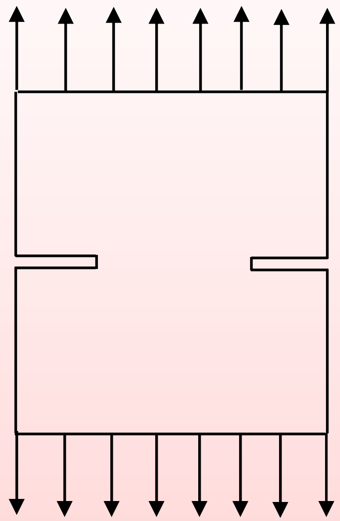


Laplace equation

$$\nabla^2(x) = 0, \quad x \in D$$

∇^2 : **Laplace operator**

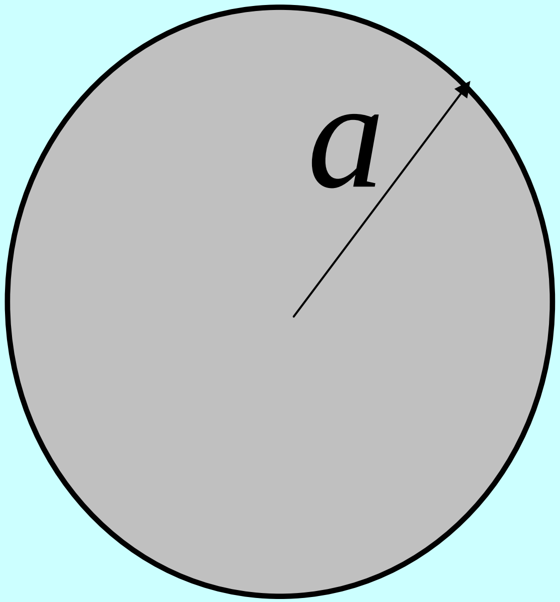
D : **domain of interest**



Treatment for degenerate boundary problems

1. cutoff wall
2. sheet pile
3. crack
4. baffle
5. thin airfoil
6. antenna

The degenerate scales for different geometry shapes



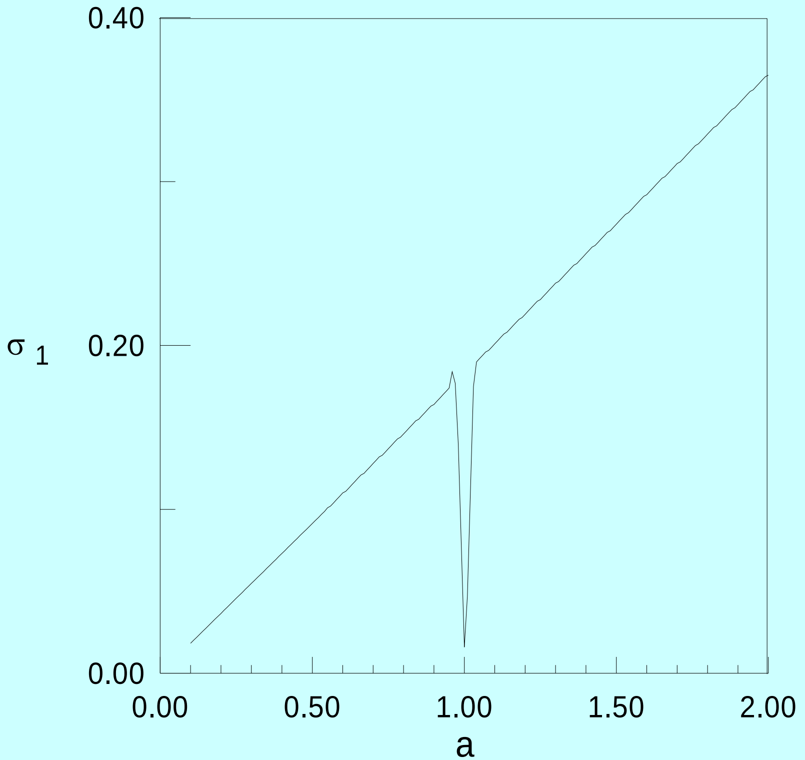


Fig.2-5 The minimum singular value σ_1 of [U] versus radius a for the interior potential problem with a circular domain.

Conclusions

- The theory of dual integral equation has been reviewed
- The role of hypersingularity is examined
- The applications of dual BEM to Helmholtz equation have been demonstrated.
- The applications of dual BEM to Laplace equation have been demonstrated.