Part II

Acoustic eigenproblem

Outlines

1. Introduction
2. 2-D acoustic eigenproblem
3. SVD updating terms
4. SVD updating documents
5. 3-D cavity
6. Conclusions

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Eigenproblem

$$u''(x) = \lambda u(x)$$

$$\nabla^2 u(x) = \lambda u(x)$$

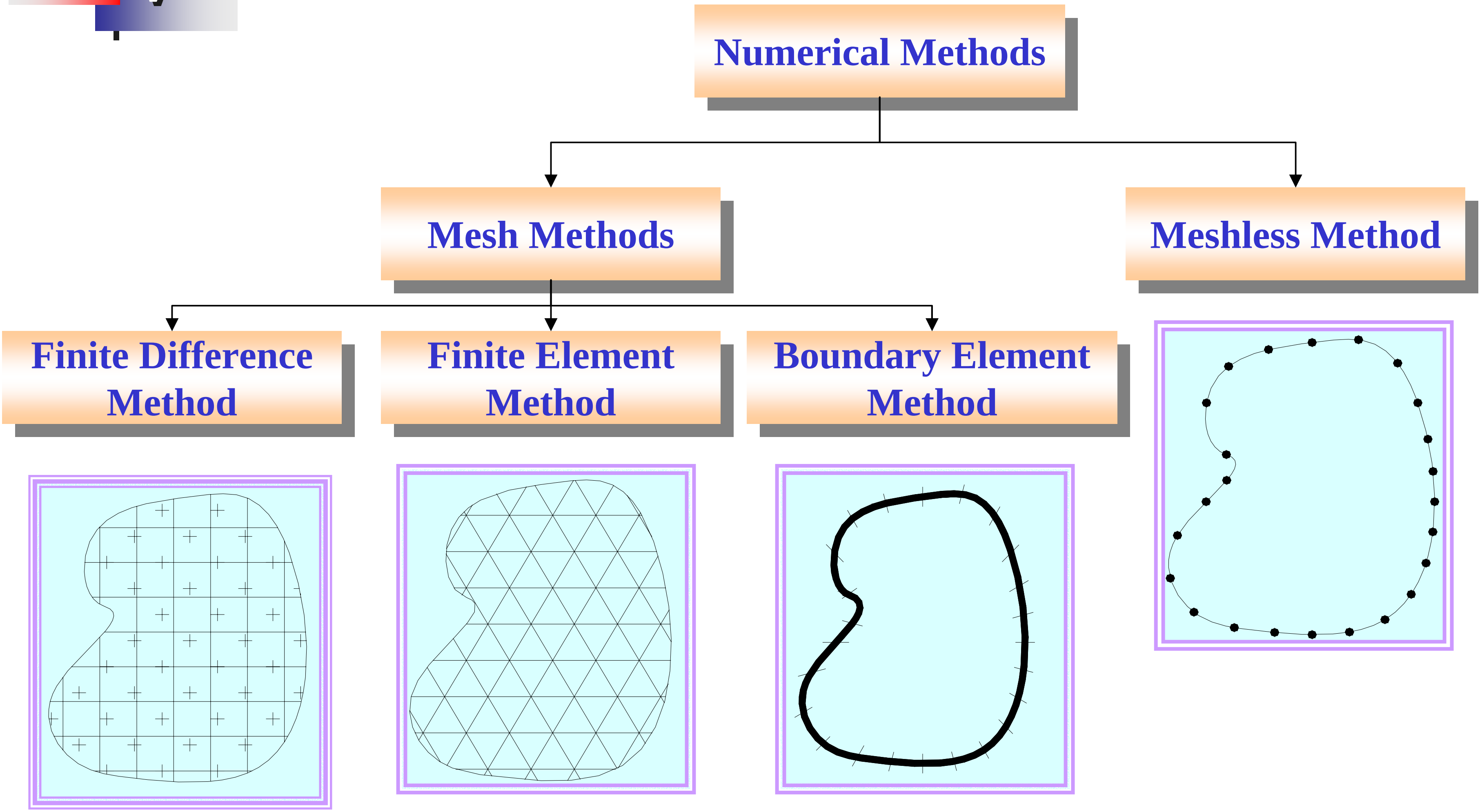
$$\nabla^4 u(x) = \lambda u(x)$$

$$\int K(x,s)u(s)ds = \lambda u(x)$$

$$[A]\{x\} = \lambda \{x\}$$

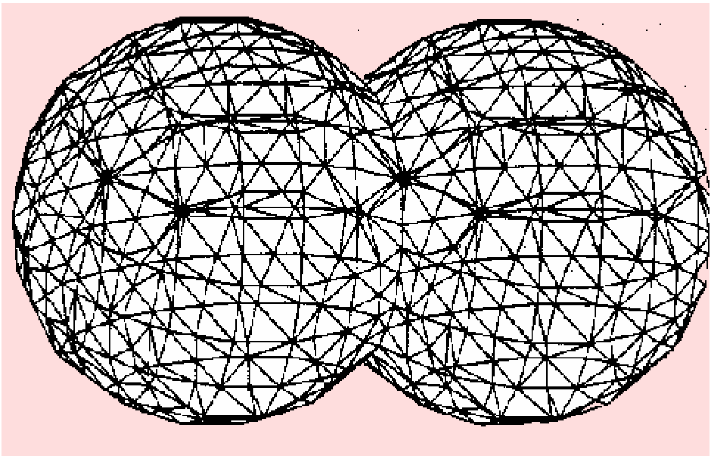
$$[K(\lambda)]\{x\} = 0$$

Numerical methods

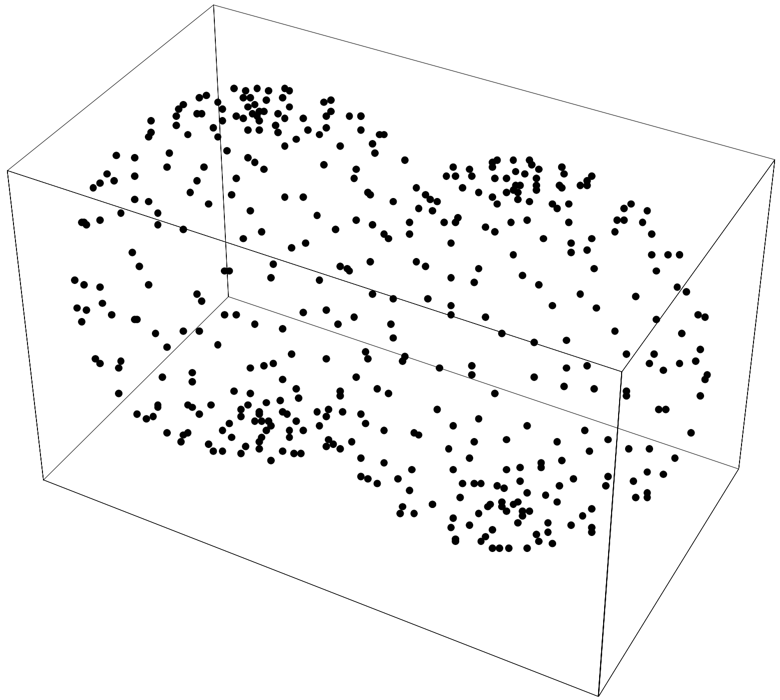


What is meshless method ?

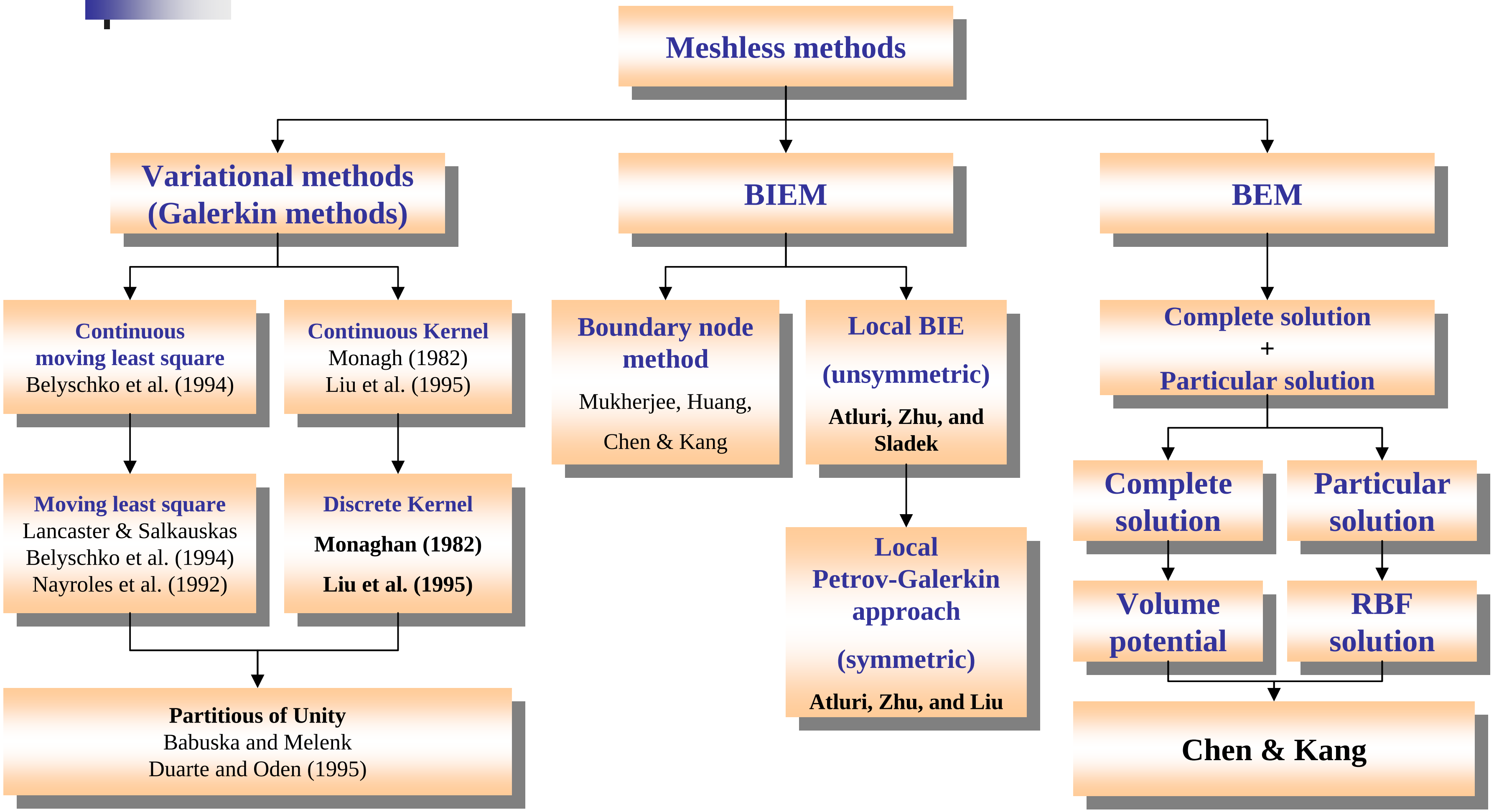
MESH METHOD
(BEM or FEM)



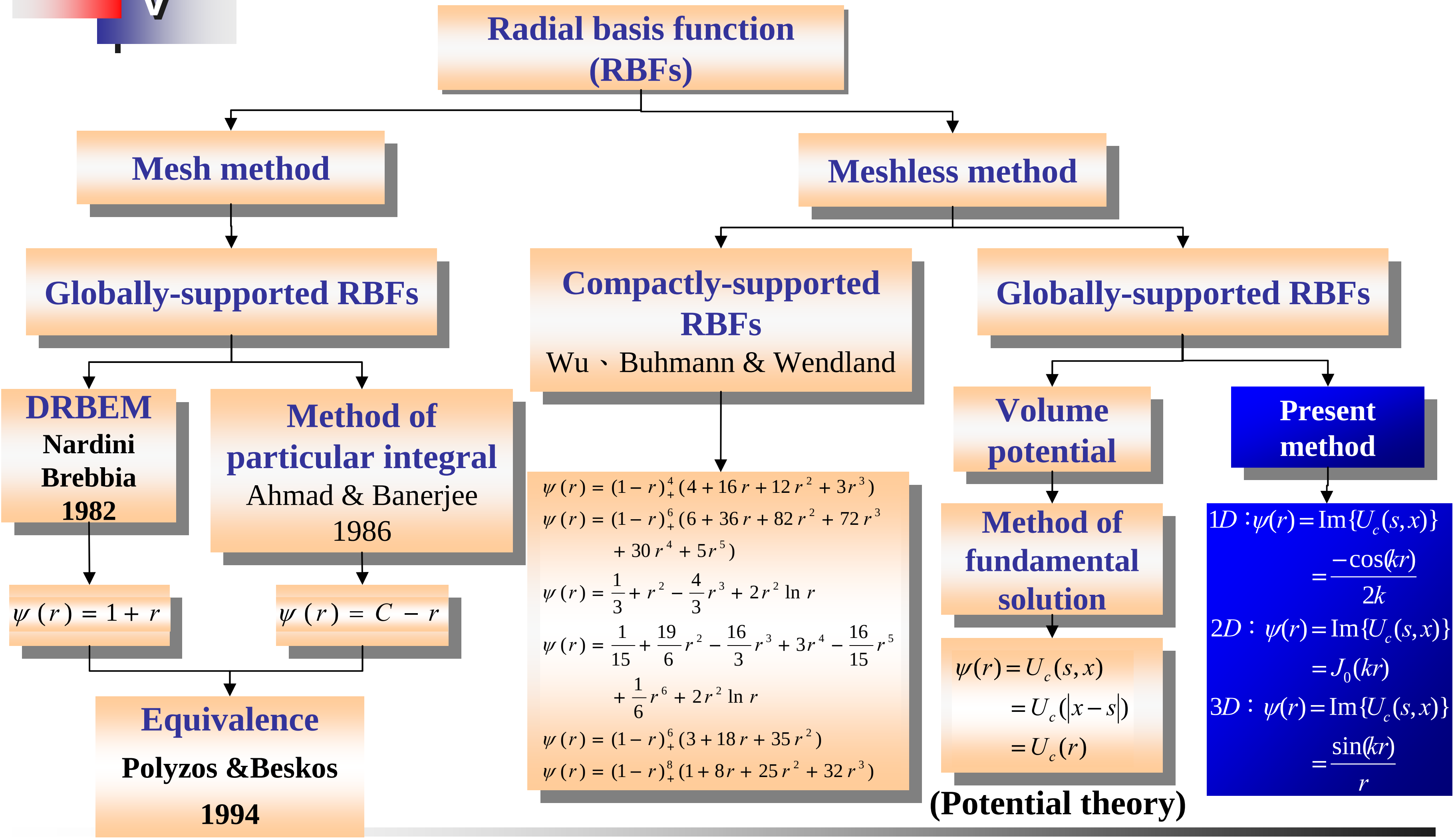
MESHLESS METHOD



The developments of meshless methods



The developments of radial basis functions



Introduction of Radial Basis Functions

a. What is radial basis function?

$$\varphi(r) = \varphi\|x - x_k\|,$$

where x is the field point and x_k is the interpolation point.

b. Scattered data interpolation (Software: Surfer, Grapher…)

$$g(x) = \sum_{k=1}^{N} c_k \varphi(\|x\text{-}x_k\|)\text{; where } \varphi(\|x\text{-}x_k\|) \text{ is radial basis function.}$$

c. This problem could lead to a linear system $\mathrm{Ac} = \mathrm{f}.$

$$A_{jk} = \varphi(\|x_j\text{-}x_k\|), \quad j,k = 1, 2, \cdots, n$$

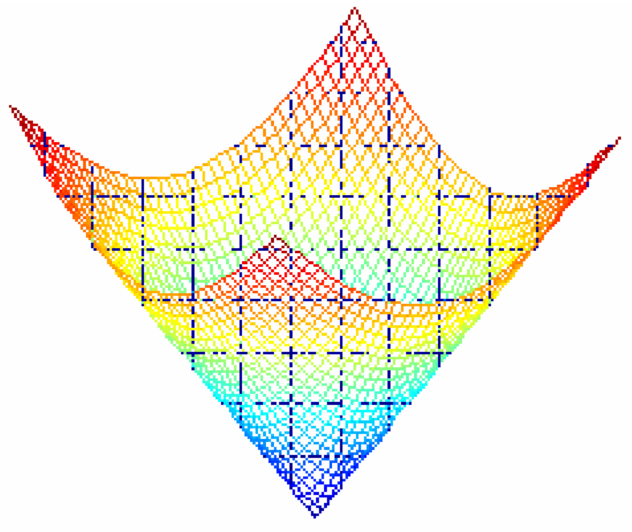
Globally Supported Radial Basis Functions

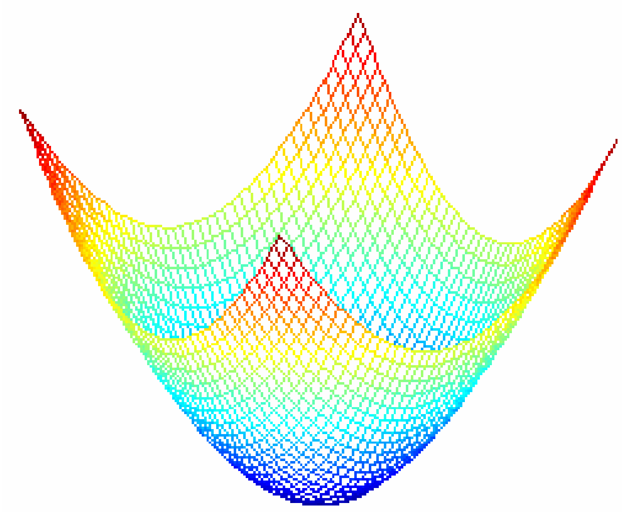
a. Globally Supported Radial Basis Functions (GSRBF)

$$\varphi = 1 + r$$

(DRBEM or Method of Particular Integrals)

$$\varphi = \sqrt{r^2 + c^2}$$

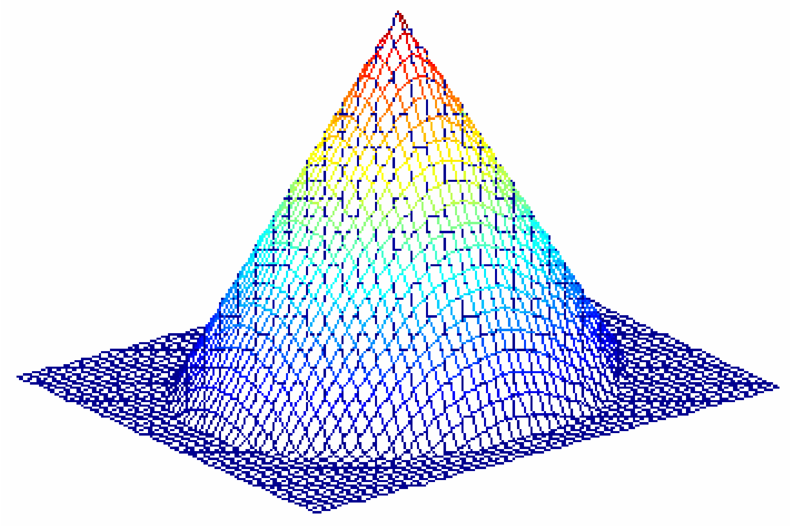


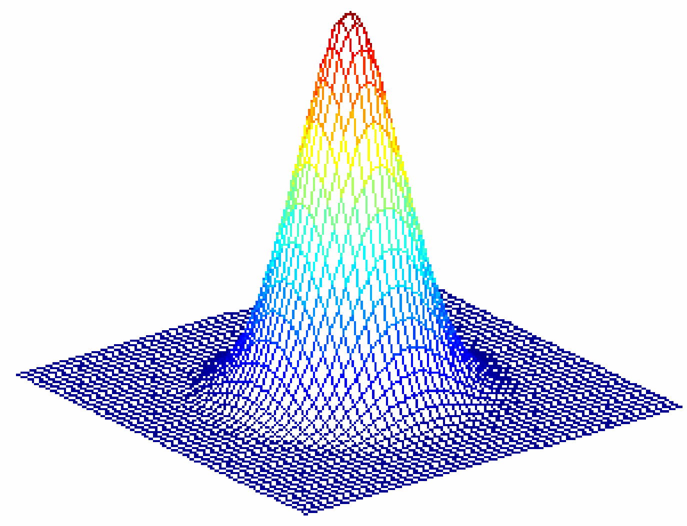


Compactly Supported Radial Basis Functions

b. Compactly Supported Radial Basis Functions (CSRBF)

$$\varphi = (1-r)_+^2 \qquad \varphi = (1-r)_+^4(4r+1)$$





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RBF in the MFS

Governing Eq.: $(\nabla^2 + k^2)u(x) = 0$

One-dimension	Two-dimension	Three-dimension
$\mathrm{Im}\left(\dfrac{-ie^{ikr}}{2k}\right)$	$\mathrm{Im}\left(\dfrac{-ikH_0(kr)}{2}\right)$	$\mathrm{Im}\left(\dfrac{e^{ikr}}{r}\right)$
$\dfrac{-\cos(kr)}{2k}$	$-J_0(kr)$	$\dfrac{\sin(kr)}{r}$

The derivation of indeterminate forms

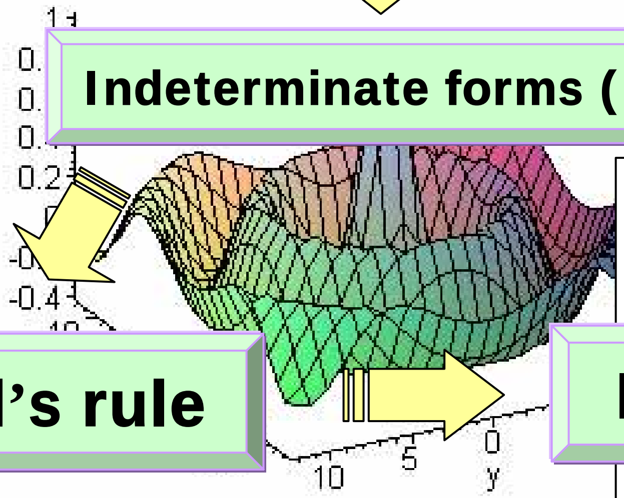
RBF for $J_0(kr)$

↓

Influence matrices

↓

Indeterminate forms ($\frac{0}{0}$)



L'Hôpital's rule → **Invariant method**

$$\lim_{s\to x} U(s,x) = 1$$

$$\lim_{s\to x} T(s,x) = 0$$

$$\lim_{s\to x} M(s,x) = \frac{k^2}{2}$$

The diagonal elements of *U* kernel

$[U(s,x)]$ ⇒

$$\lambda_\ell = 2NJ_\ell(k\ell)J_\ell(k\ell), \quad \ell = 0, \pm 1, \cdots, \pm(N-1), N.$$

$$2Na_0 = 2N\sum_{\ell=-\infty}^{\ell=\infty} J_\ell(k\ell)J_\ell(k\ell).$$

+

$$1 = J_0^2(k\ell) + 2\sum_{m=-\infty}^{\infty} J_m^2(k\ell).$$

⇐ **Addition theorem of Bessel function**

diag $([U(s,x)])$ ⇒ $\underline{\underline{a_0 = 1.}}$

The diagonal elements of T kernel

$[T(s,x)]$ ⇒

$$\mu_\ell = 2NJ'_\ell(k\ell)J_\ell(k\ell), \quad \ell = 0, \pm 1, \cdots, \pm(N-1), N.$$

$$2Nb_0 = 2N\,k \sum_{\ell=-\infty}^{\ell=\infty} J'_\ell(k\ell)J_\ell(k\ell).$$

+

$$0 = J_0(k\ell)J'_0(k\ell) + 2\sum_{m=-\infty}^{\infty} J_m(k\ell)J'_m(k\ell).$$

⇐ **Addition theorem of Bessel function**

diag $([T(s,x)])$ ⇒ $\underline{\underline{b_0 = 0.}}$

Two-dimensional acoustic eigenproblem

RBF for $J_0(kr)$

↓

Influence matrices
U(s,x), L(s,x), T(s,x), M(s,x)
+
Indirect method

↓

Eigenvalues

↓

Eigenmodes

SVD (Singular Value Decomposition)

$$[A(\lambda)]\{x\} = 0$$

SVD → $$[A]_{m\times n} = [\Phi]_{m\times m}[\Sigma]_{m\times n}[\Psi]^T{}_{n\times n}$$

J. T. Chen, C. F. Lee and S. Y. Lin.

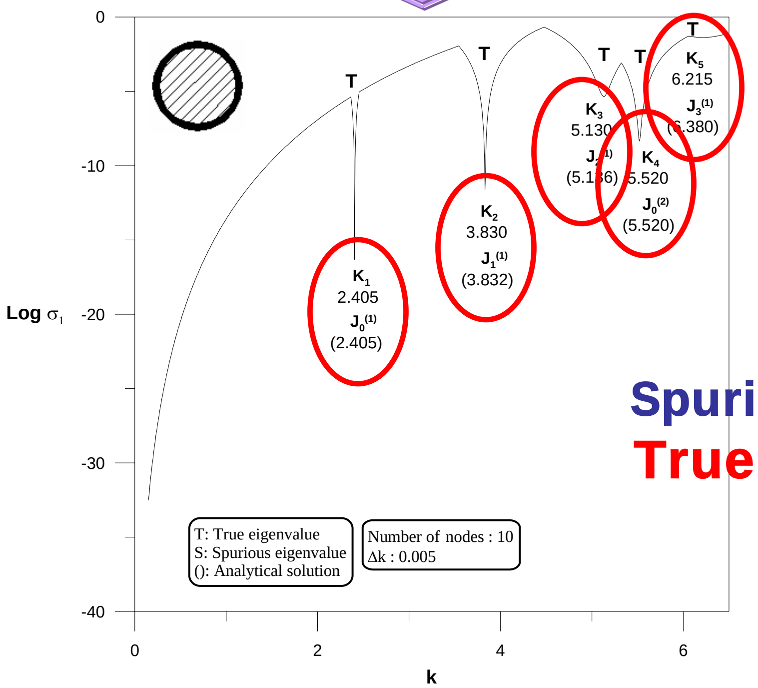
A new point of view for the polar decomposition using singular value decomposition.

International Journal of Computational and Numerical Analysis and Applications.

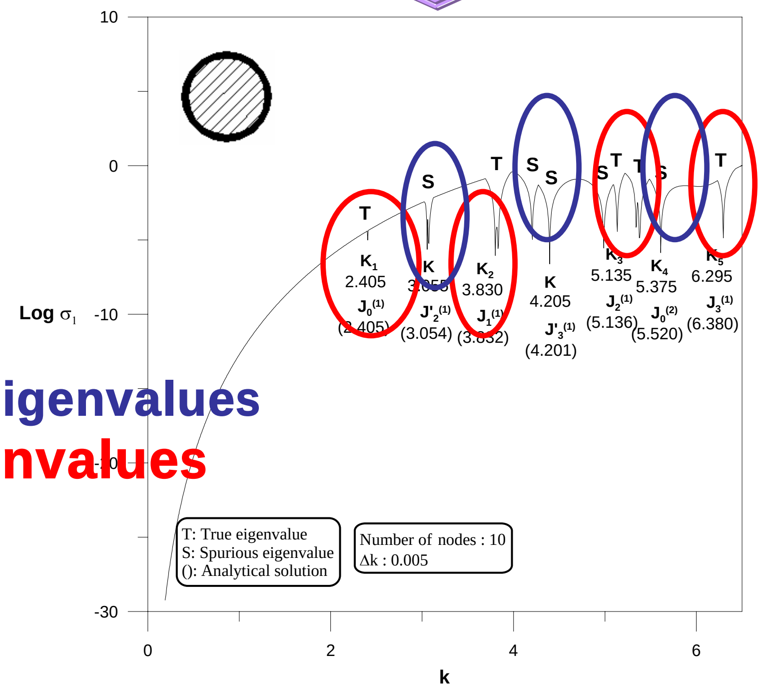
Vol.2, No.3, 257-264 , 2002.

Eigenfrequencies for Dirichlet B.C.

Single-layer potential approach



Double-layer potential approach



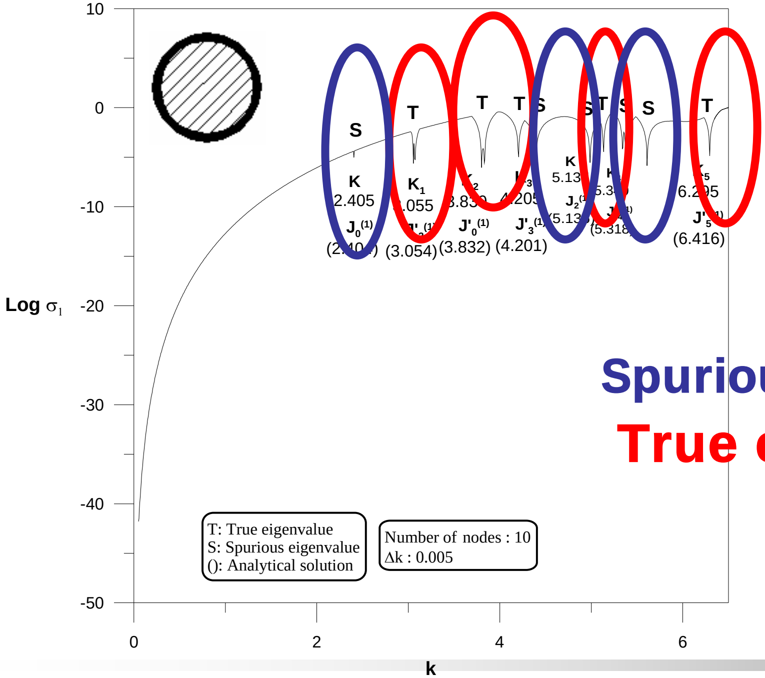
Spurious eigenvalues

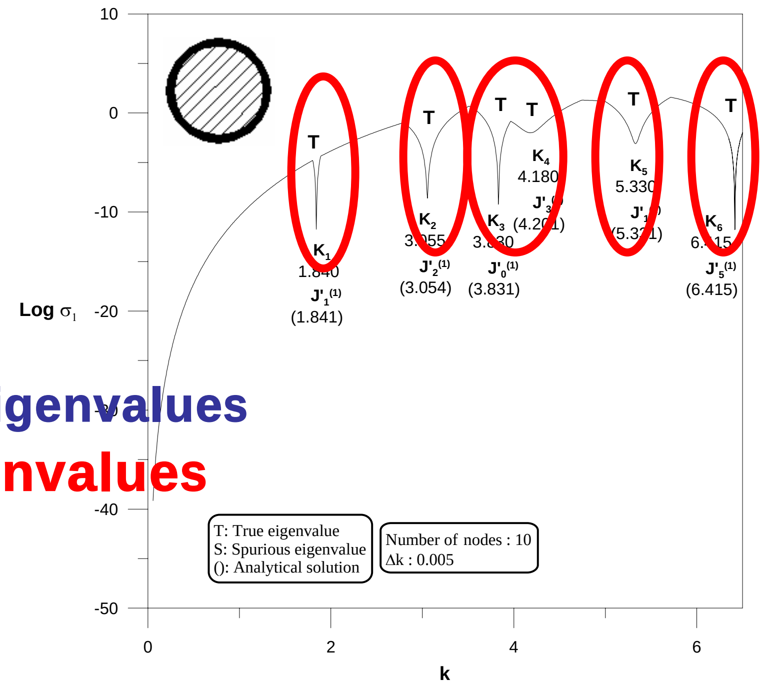
True eigenvalues

Eigenfrequencies for Neumann B.C.

Single-layer potential approach

Double-layer potential approach





Spurious eigenvalues

True eigenvalues

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SVD

Diagonal matrix

$$[\Sigma]=\begin{bmatrix}\sigma_1 & 0 & \cdots & 0\\ 0 & \sigma_2 & 0 & 0\\ 0 & 0 & \ddots & \vdots\\ 0 & 0 & \cdots & \sigma_n\\ \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}_{m\times n}$$

Unitary matrix $[\Phi]_{m\times m}$, $[\Psi]_{n\times n}$

To extract the true eigenvalues

Deficient constraints

↓

Spurious eigensolutions

↓ ← **Additional constraints**

SVD
updating terms

↓

To extract the true eigensolutions

SVD updating terms

Direct method for Dirichlet B. C. :

Singular equation (UT method)

$$[T^E]\,\underset{\sim}{u} = [U^E]\,\underset{\sim}{t} = 0.$$

Hypersingular equation (LM method)

$$[M^E]\,\underset{\sim}{u} = [L^E]\,\underset{\sim}{t} = 0.$$

$$\underset{\sim}{t} = \{\psi_j\}$$

$$\begin{bmatrix} U^E \\ L^E \end{bmatrix}\{\psi_j\} = 0$$

↓

SVD updating terms

To extract the true eigenfrequencies

SVD updating terms

$$[U^E]\{\psi_i\}=\{0\}$$

$$[L^E]\{\psi_i\}=\{0\}$$

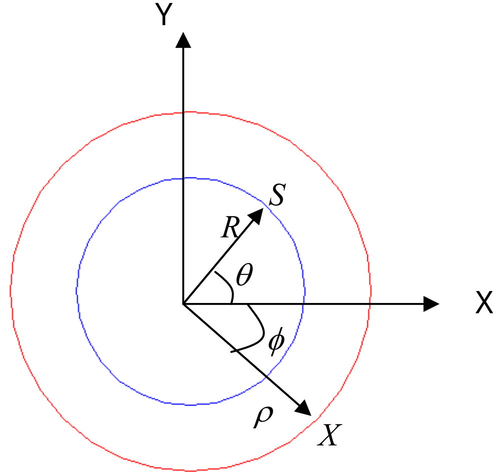
$$\sum_j \sigma_j^U\{\phi_j^U\}\{\psi_j^U\}^T\{\psi_i\}=\{0\} \xrightarrow[\psi_j^U\cdot\psi_j=\delta_{ij}]{} \sigma_j^U\{\phi_j^U\}=\{0\}$$

$$\sum_j \sigma_j^L\{\phi_j^L\}\{\psi_j^L\}^T\{\psi_i\}=\{0\} \xrightarrow[\psi_j^L\cdot\psi_j=\delta_{ij}]{} \sigma_j^L\{\phi_j^L\}=\{0\}$$

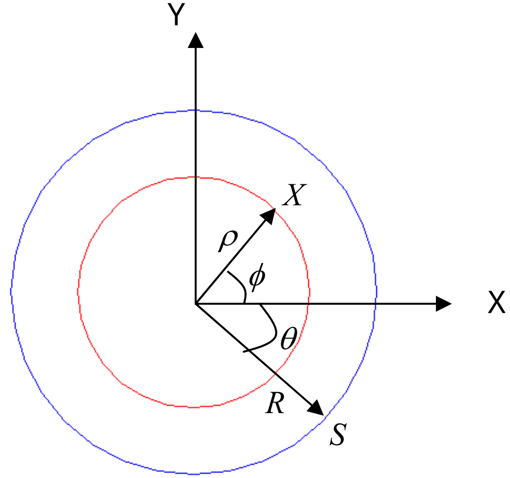
$$\sigma_i^U=\sigma_i^L=0$$

The degenerate kernels

The degenerate kernels for interior and exterior problems:



Interior problem



Exterior problem

Blue: field points

Red: source points

$$U(s,x)=\begin{cases} U^{I}(R,\theta;\ \rho,\phi)=-\dfrac{\pi}{2}\displaystyle\sum_{n=-\infty}^{\infty}J_m(kR)J_m(k\rho)\big(\cos(m(\theta-\phi))\big), & R>\rho \\ U^{E}(R,\theta;\ \rho,\phi)=-\dfrac{\pi}{2}\displaystyle\sum_{n=-\infty}^{\infty}J_m(k\rho)J_m(kR)\big(\cos(m(\theta-\phi))\big), & R<\rho \end{cases}$$

$$U^{E}=U^{I}$$

The degenerate kernels

$$T(s,x)=\begin{cases} T^I(R,\theta;\rho,\phi)=-\dfrac{k\pi}{2}\displaystyle\sum_{n=-\infty}^{\infty}J_m(k\rho)J'_m(kR)(\cos(m(\theta-\phi))), & R>\rho \\ T^E(R,\theta;\rho,\phi)=-\dfrac{k\pi}{2}\displaystyle\sum_{n=-\infty}^{\infty}J_m(kR)J'_m(k\rho)(\cos(m(\theta-\phi))), & R<\rho \end{cases}$$

$$L^E=T^I$$

$$L(s,x)=\begin{cases} L^I(R,\theta;\rho,\phi)=-\dfrac{k\pi}{2}\displaystyle\sum_{n=-\infty}^{\infty}J'_m(k\rho)J_m(kR)(\cos(m(\theta-\phi))), & R>\rho \\ L^E(R,\theta;\rho,\phi)=-\dfrac{k\pi}{2}\displaystyle\sum_{n=-\infty}^{\infty}J'_m(kR)J_m(k\rho)(\cos(m(\theta-\phi))), & R<\rho \end{cases}$$

$$\begin{bmatrix} U^E \\ L^E \end{bmatrix} \xrightarrow[L^E=T^I]{U^E=U^I} \begin{bmatrix} U^I \\ T^I \end{bmatrix}$$

SVD techniques for eigensolutions

Boundary Value Problem	Eigen-solution	Density function		True and spurious eigenvalues			
		Single-layer potential	Double layer potential	Direct method		Indirect method	
Dirichlet Problem	True	$J_m(k\rho)=0$	$J_m(k\rho)=0$	SVD updating term	$\begin{bmatrix} U^E \\ L^E \end{bmatrix}$	SVD updating term	$\begin{bmatrix} U^I \\ T^I \end{bmatrix}$
	Spurious	$J'_m(k\rho)=0$	$J_m(k\rho)=0$	SVD updating document	$\begin{bmatrix} L^E & M^E \end{bmatrix}$	SVD updating document	$\begin{bmatrix} T^I & M^I \end{bmatrix}$
Neumann Problem	True	$J'_m(k\rho)=0$	$J'_m(k\rho)=0$	SVD Updating term	$\begin{bmatrix} T^E \\ M^E \end{bmatrix}$	SVD updating term	$\begin{bmatrix} L^I \\ M^I \end{bmatrix}$
	Spurious	$J_m(k\rho)=0$	$J'_m(k\rho)=0$	SVD updating document	$\begin{bmatrix} U^E & T^E \end{bmatrix}$	SVD updating document	$\begin{bmatrix} U^I & L^I \end{bmatrix}$

To extract the true eigenvalues by the indirect method

Indirect method for Dirichlet B. C. :

$$\begin{bmatrix} U^I \\ T^I \end{bmatrix} \{\psi_j\} = 0$$

↓

SVD
updating terms

↓

$$\sum_j \sigma_j^U \{\phi_j^U\}\{\psi_j^U\}^T \{\psi_i\} = \{0\} \xrightarrow[\psi_j^U \cdot \psi_j = \delta_{ij}]{} \sigma_j^U \{\phi_j^U\} = \{0\}$$

$$\sum_j \sigma_j^T \{\phi_j^T\}\{\psi_j^T\}^T \{\psi_i\} = \{0\} \xrightarrow[\psi_j^T \cdot \psi_j = \delta_{ij}]{} \sigma_j^T \{\phi_j^T\} = \{0\}$$

↓

Singular-layer approach: $J_i(k\rho) J_i(k\rho) = 0, \quad (i \text{ no sum})$

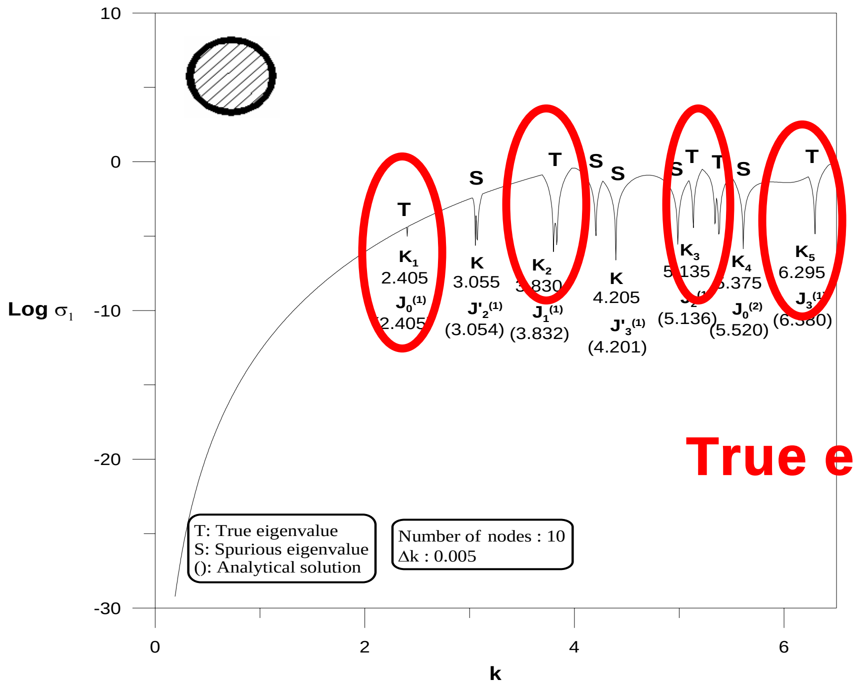
Double-layer approach: $J_i(k\rho) J_i'(k\rho) = 0, \quad (i \text{ no sum})$

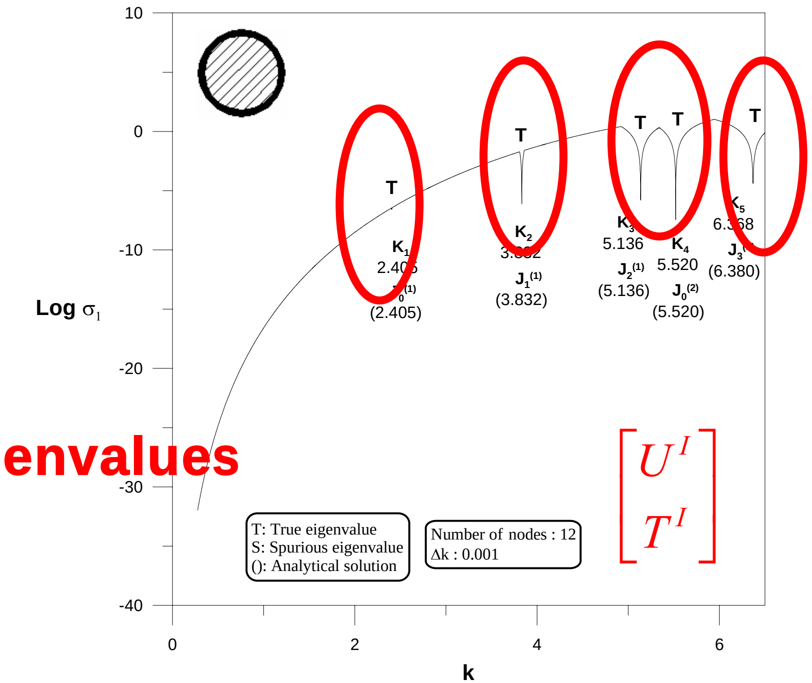
Examples for the SVD updating terms

For Dirichlet B.C.

Double-layer potential approach

SVD updating-terms





True eigenvalues

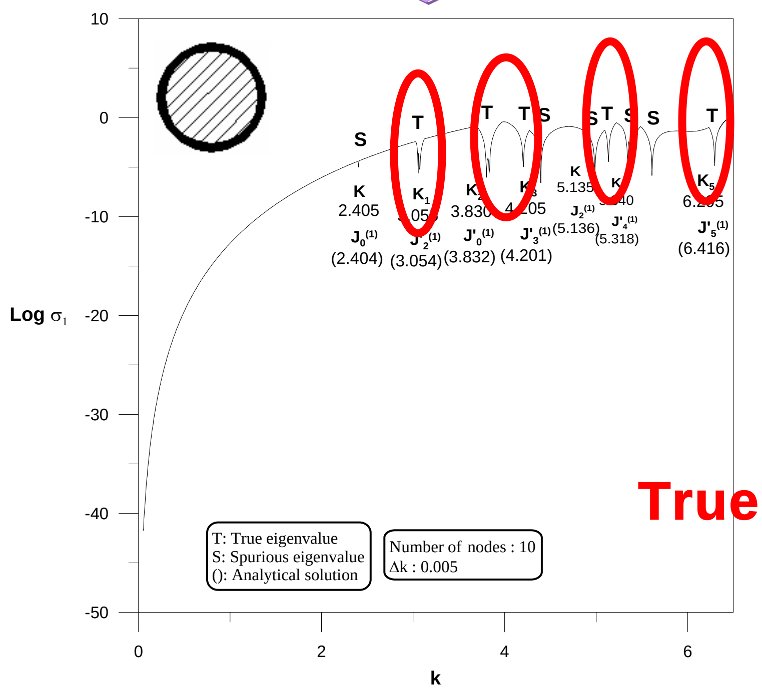
$$\begin{bmatrix} U^I \\ T^I \end{bmatrix}$$

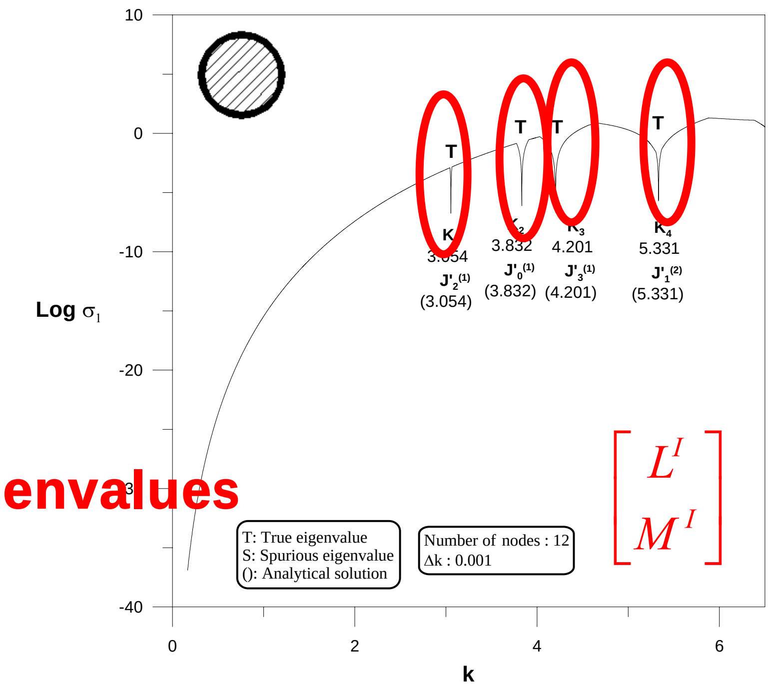
Examples for the SVD updating terms

For Neumann B.C.

Single-layer potential approach

SVD updating-terms





True eigenvalues

$$\begin{bmatrix} L^I \\ M^I \end{bmatrix}$$

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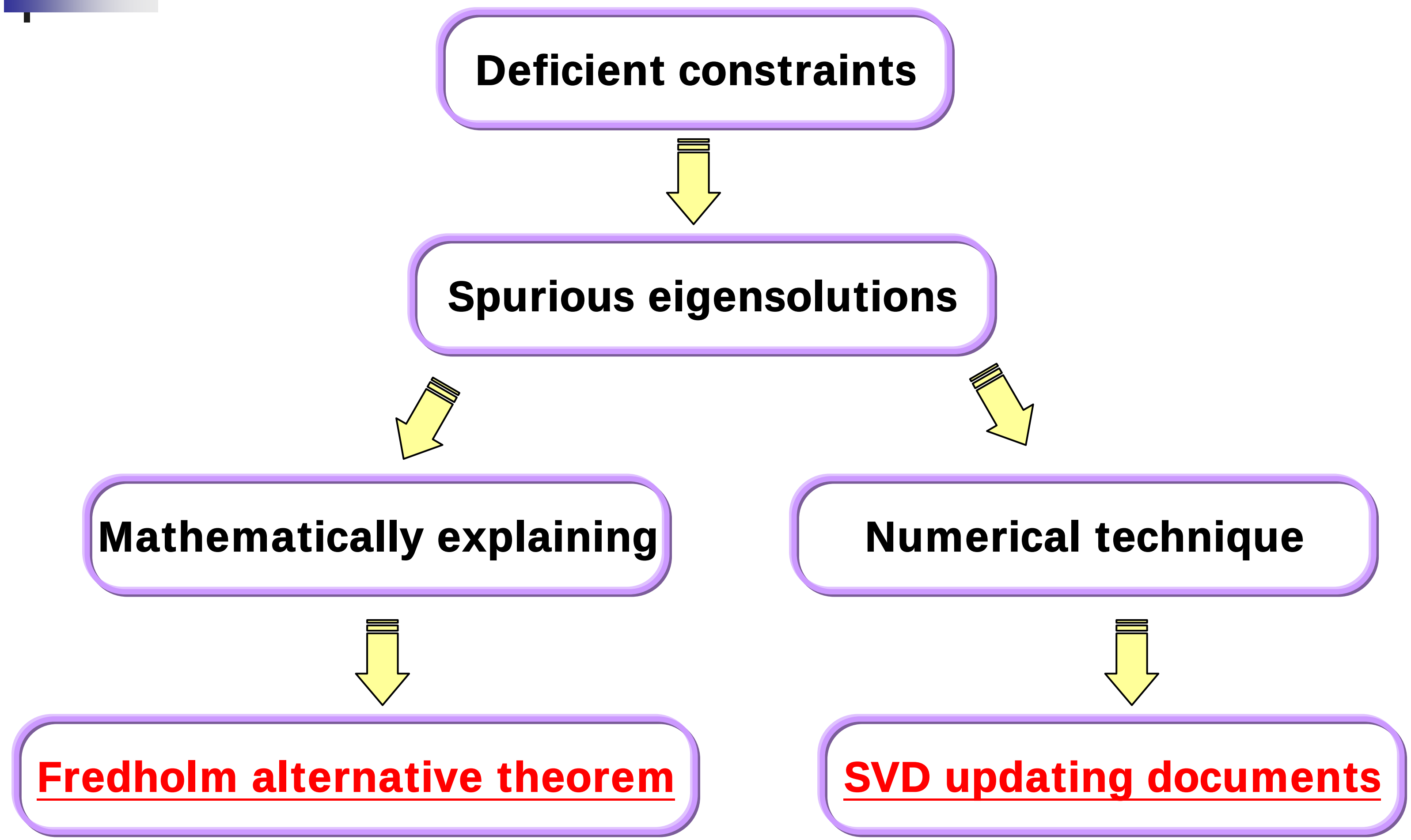
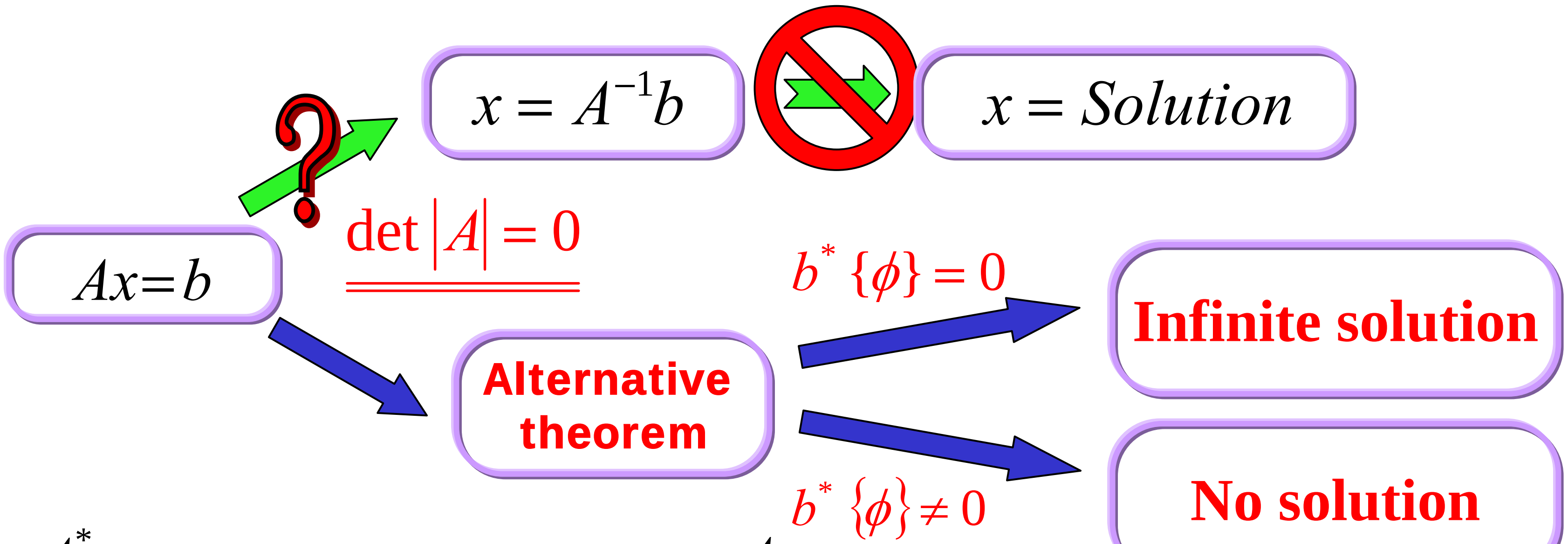
To filter out the spurious solutions



Fredholm alternative theorem

Fredholm's alternative theorem:

For solving an algebraic system: $Ax=b$



A^* : the transpose conjugate matrix of A

$A^*=A^T$ if A is real

where ↗ satisfies $A^*\{\phi\}=0$

SVD updating documents

For double-layer potential approach:

$$\underbrace{\begin{Bmatrix} u(x) \\ t(x) \end{Bmatrix}}_{b} = \underbrace{\begin{bmatrix} [T(s,x)] \\ [M(s,x)] \end{bmatrix}}_{A} \underbrace{\{\psi\}}_{x}$$

$$u(x) = [T(s,x)]\{\psi\}$$

$$t(x) = [M(s,x)]\{\psi\}$$

↓

$$A^T\{\phi\} = 0 \quad \text{or} \quad \{\phi\}^T A = 0$$

↓

$$\begin{bmatrix} [T]^T \\ [M]^T \end{bmatrix}\{\phi\} = 0 \quad \text{or} \quad \underline{\underline{\{\phi\}^T\big[[T] \quad [M]\big] = 0}}$$

SVD updating documents

For double-layer potential approach:

SVD updating documents

$$\sum_j \sigma_j^T \{\psi_j^T\}\{\phi_j^T\}^T\{\phi_i\} = \{0\} \xrightarrow{\{\phi_j^T\}^T \cdot \phi_j = \delta_{ij}} \sigma_j^T \{\psi_j^T\} = \{0\}$$

$$\sum_j \sigma_j^M \{\psi_j^M\}\{\phi_j^M\}^T\{\phi_i\} = \{0\} \xrightarrow{\{\phi_j^M\}^T \cdot \phi_j = \delta_{ij}} \sigma_j^M \{\psi_j^M\} = \{0\}$$

$$\Phi\Sigma\Psi^T$$

Spurious modes

$$\sigma_i^T = \sigma_i^M = 0$$

Dirichlet problem: $J_i(k\rho)J_i'(k\rho) = 0, \quad (i \text{ no sum})$

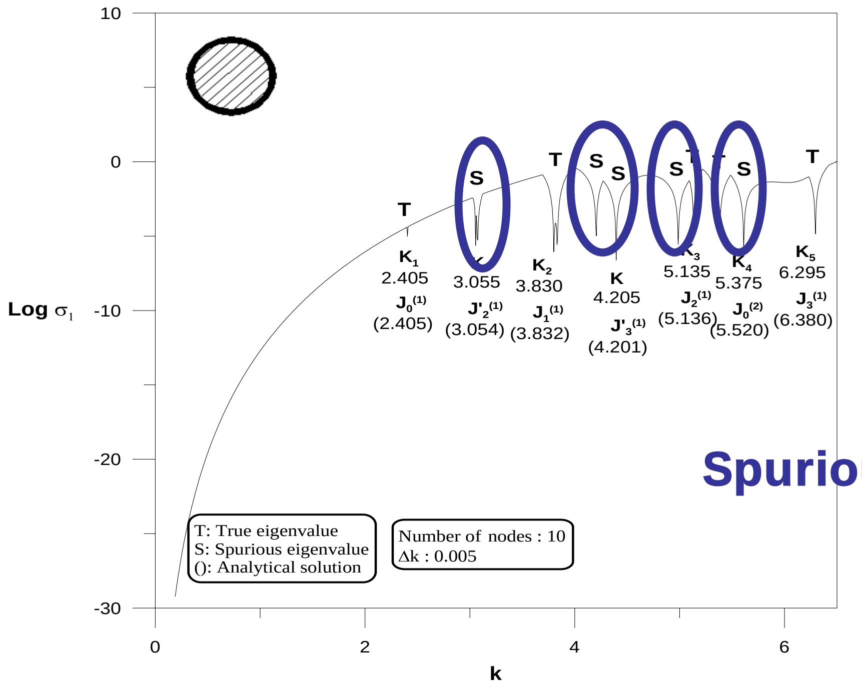
Neumann problem: $J_i'(k\rho)J_i'(k\rho) = 0, \quad (i \text{ no sum})$

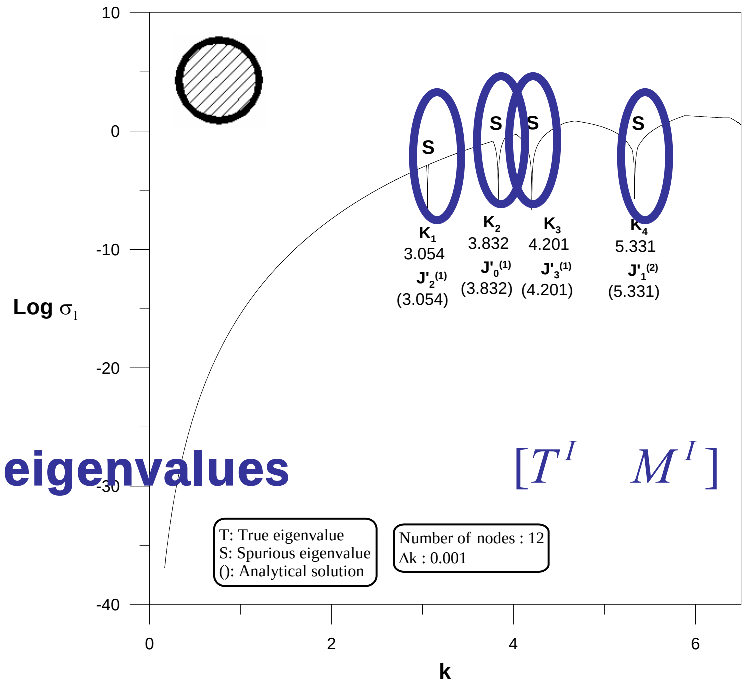
Examples for the SVD updating documents

For Dirichlet B.C.

Double-layer potential approach

SVD updating documents





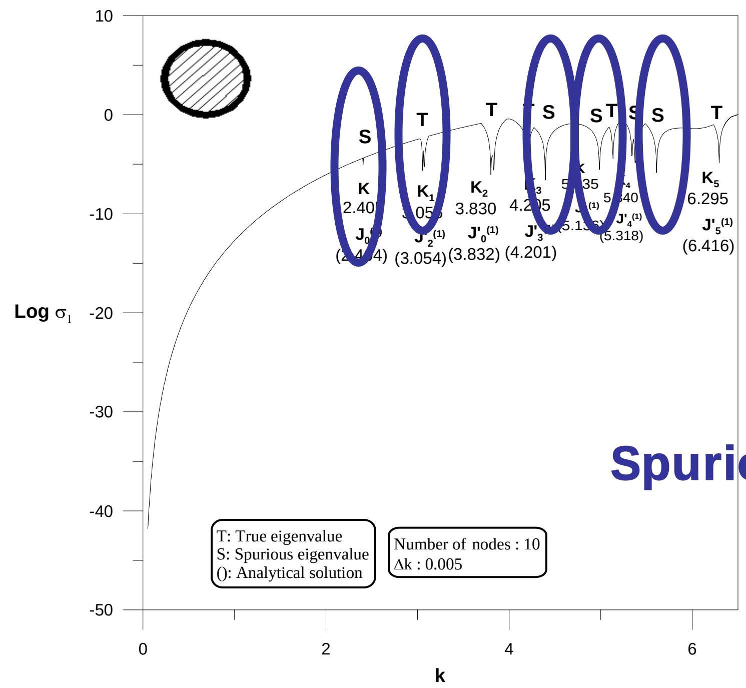
Spurious eigenvalues

$[T^I \quad M^I]$

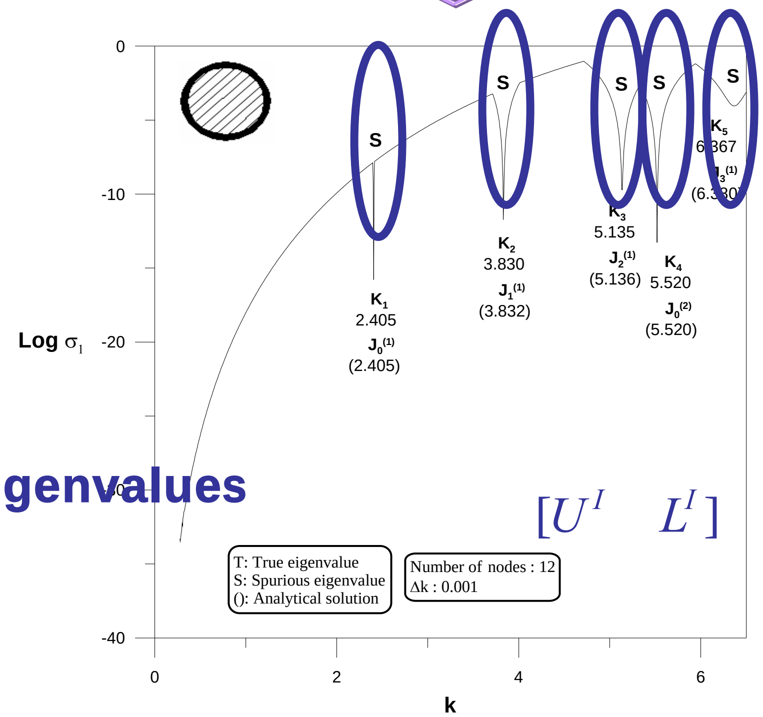
Examples for the SVD updating documents

For Dirichlet B.C.

Double-layer potential approach



SVD updating documents



Spurious eigenvalues

$$[U^I \quad L^I]$$

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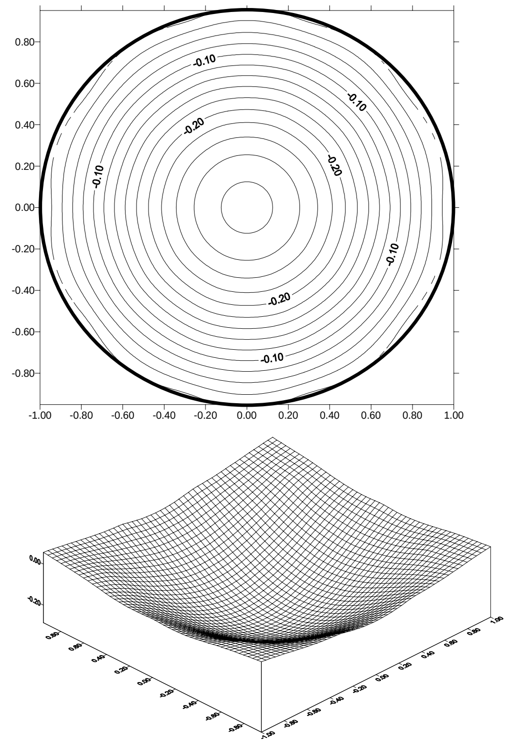
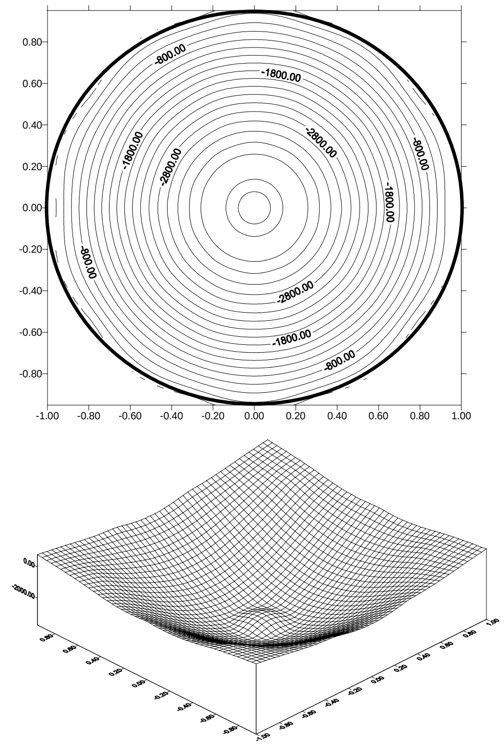
The interior modes of a circular acoustics

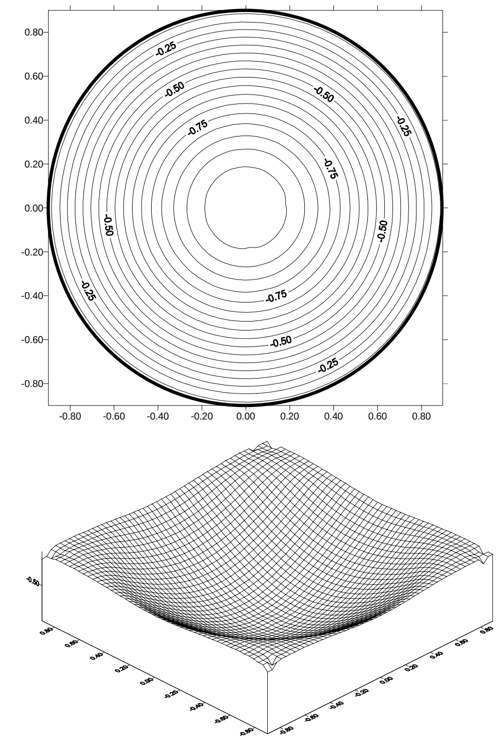
Single-layer potential approach



Double-layer potential approach



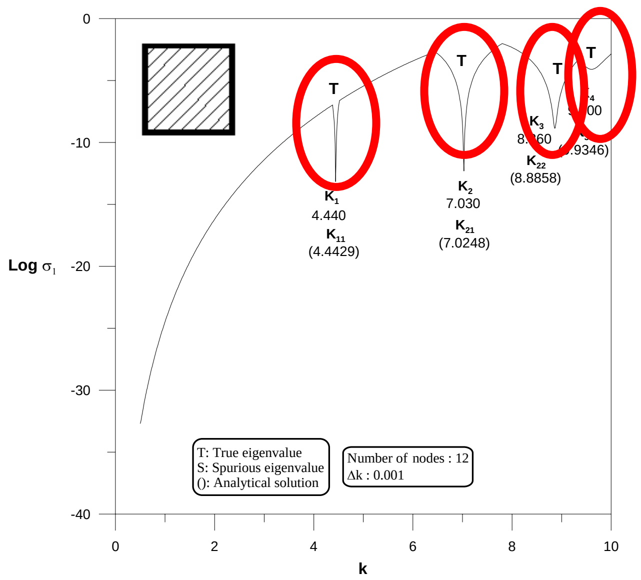
Analytical solution

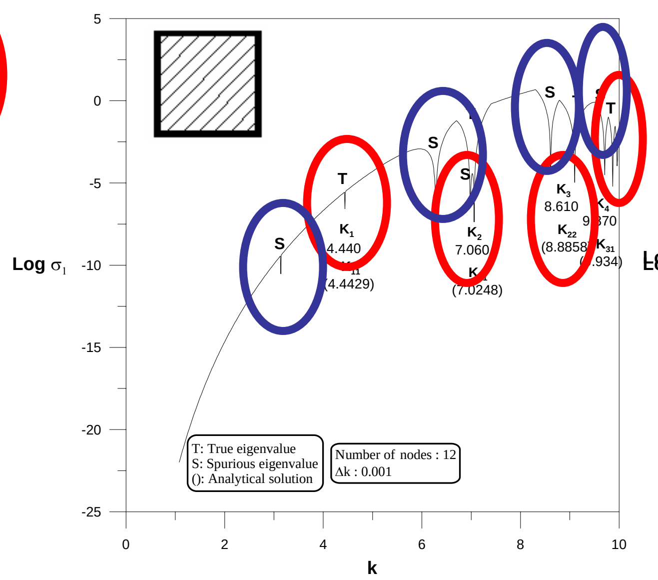


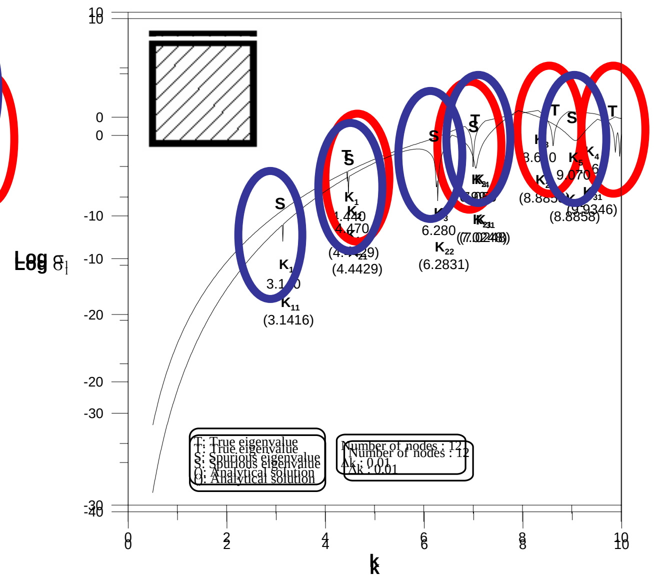
SVD techniques for a square acoustics

For Dirichlet B.C.

Single-layer potential approach	Double-layer potential approach	SVD updating terms / updating documents

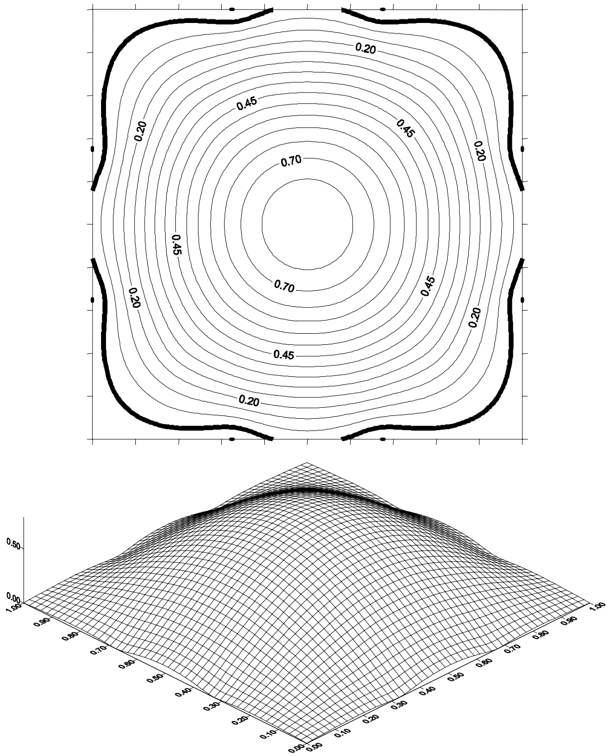
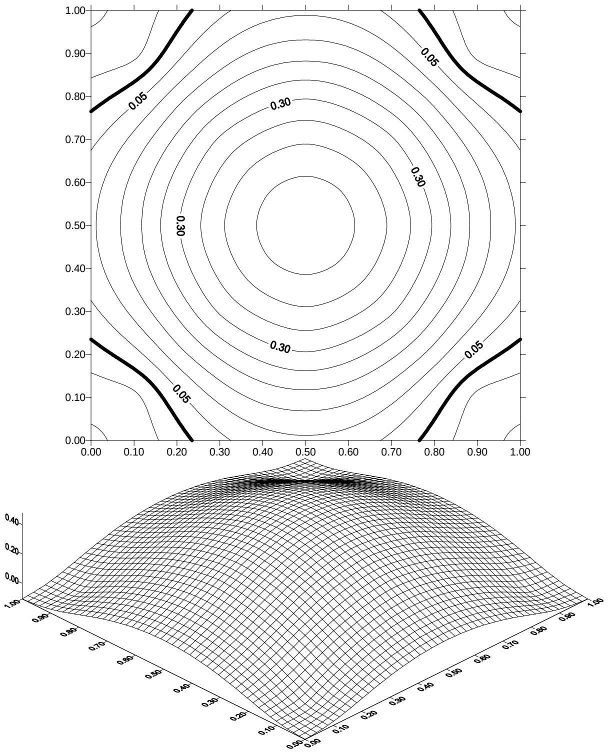
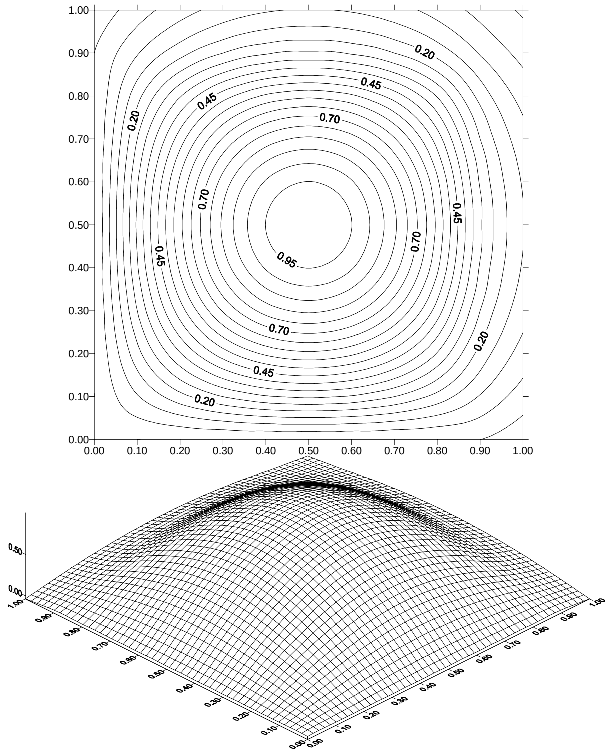






True eigenvalues / Spurious eigenvalues

The interior modes of square acoustics

Single-layer potential approach	Double-layer potential approach	Analytical solution
		

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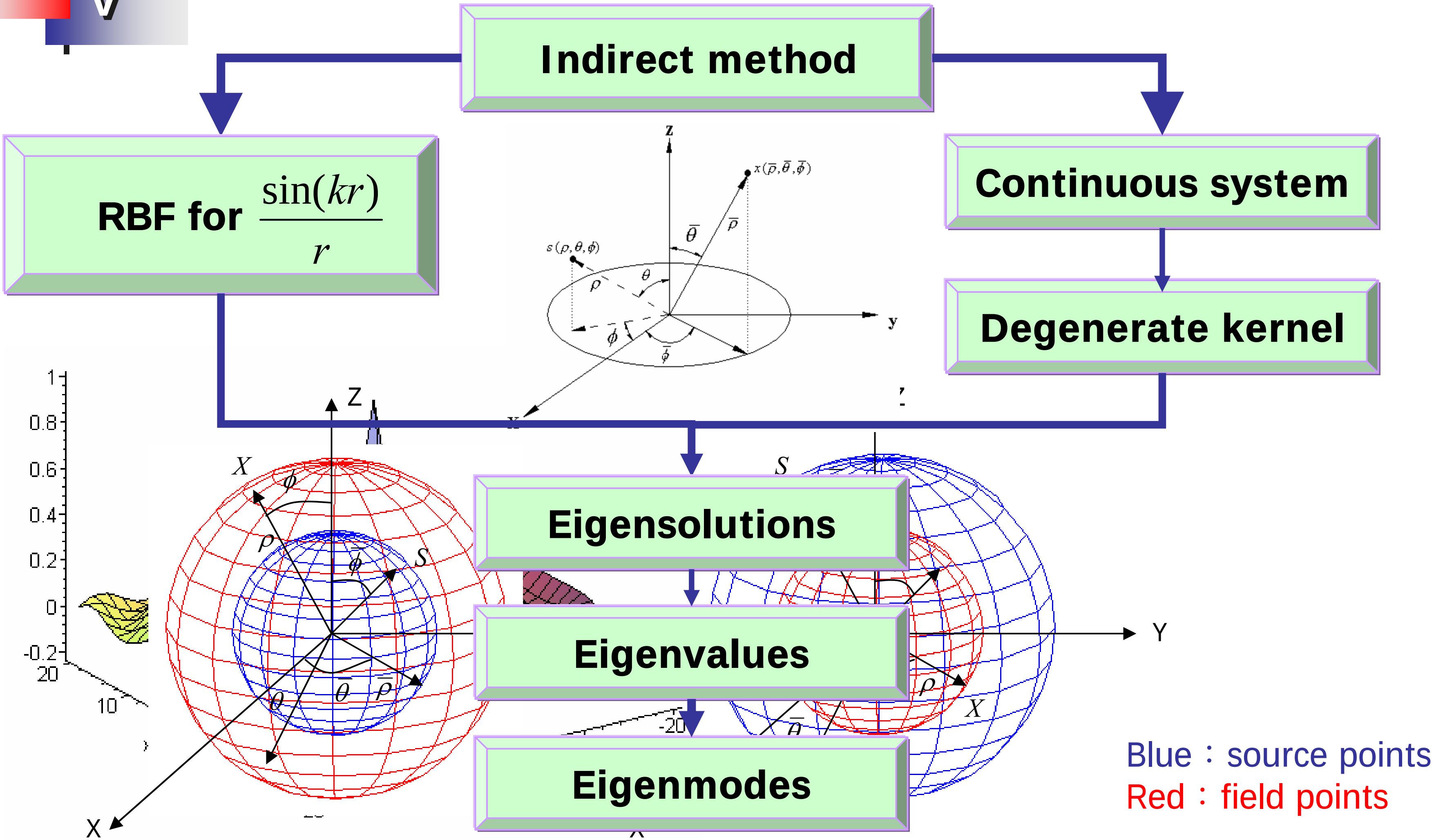
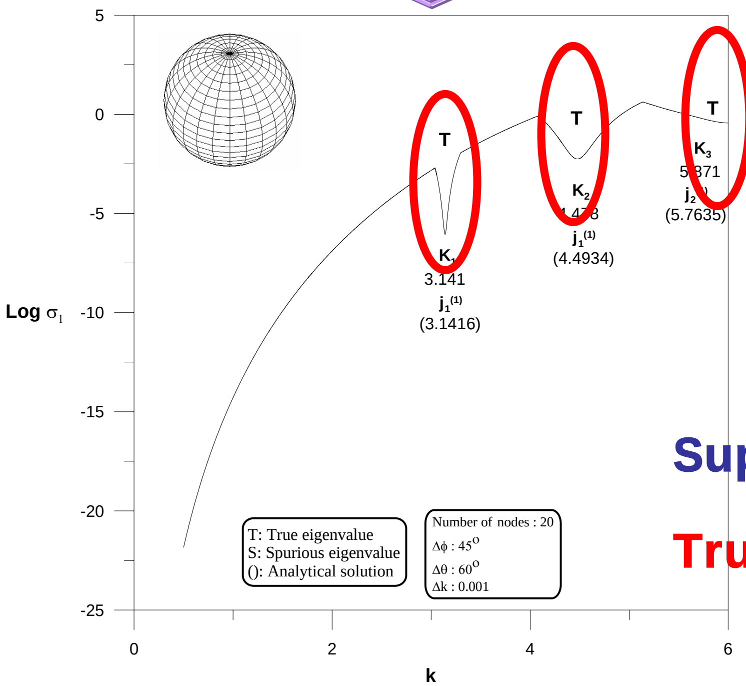
Three-dimensional spherical cavity

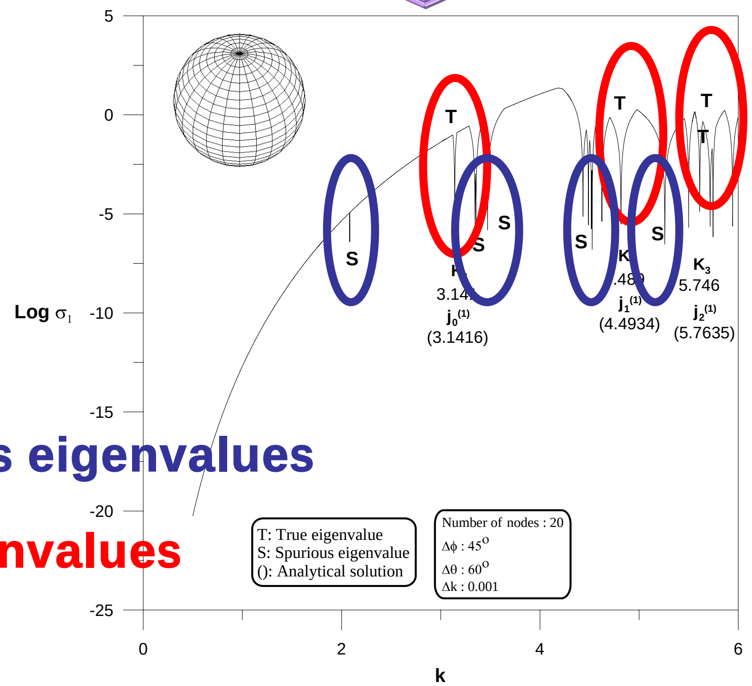


Eigenfrequencies for Dirichlet B.C.

Single-layer potential approach



Double-layer potential approach

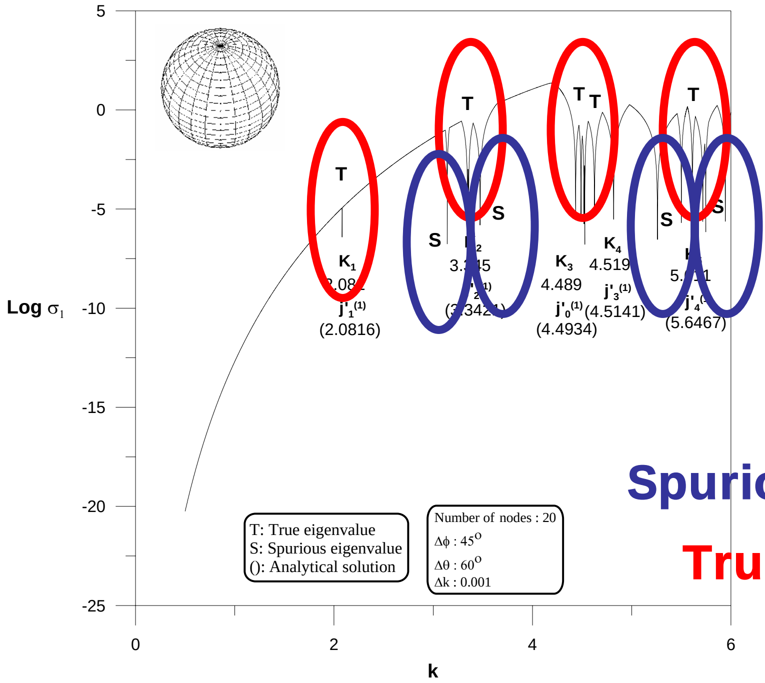


Supurious eigenvalues

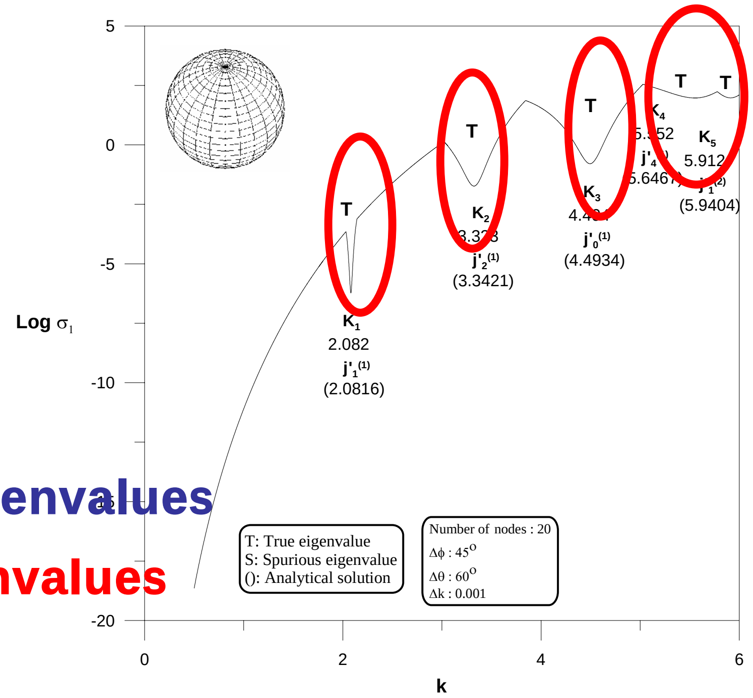
True eigenvalues

Eigenfrequencies for Neumann B.C.

Single-layer potential approach



Double-layer potential approach



Spurious eigenvalues

True eigenvalues

SVD techniques for a spherical cavity

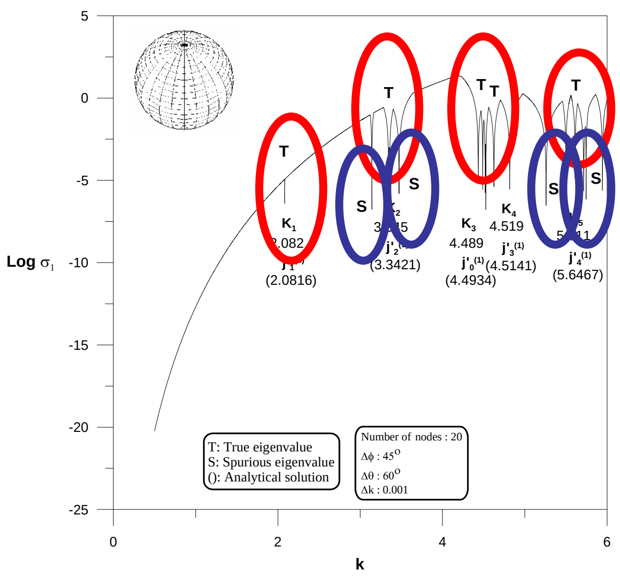
Neumann B.C.

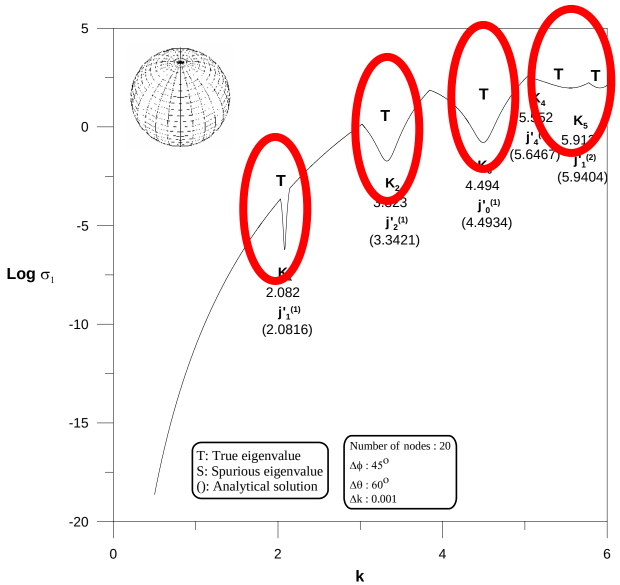
Single-layer potential approach

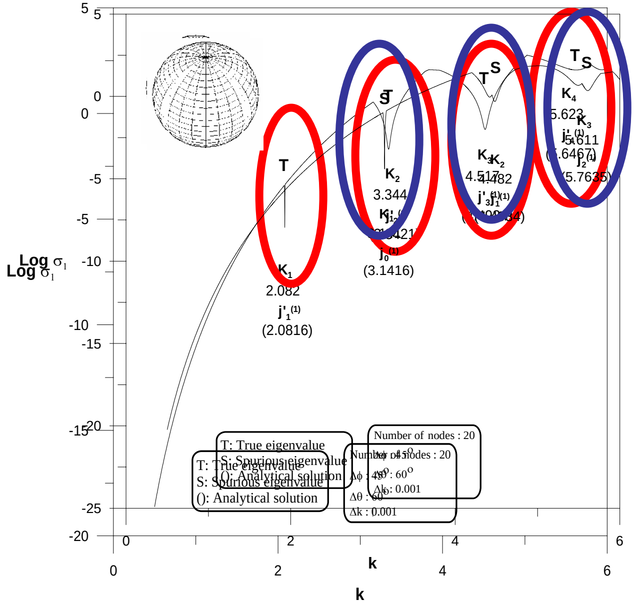
Double-layer potential approach

SVD updating documents

SVD updating terms







Spurious eigenvalues

True eigenvalues

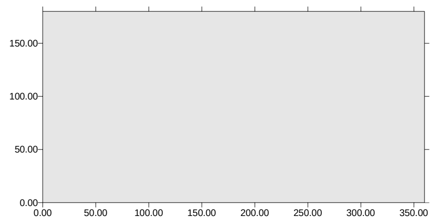
Contours for the boundary modes

$$u(1,\theta,\phi)=\begin{cases}1, & n=m=0\\ p_n^m(\cos\theta)e^{im\phi}, & \text{otherwise}\end{cases}$$

n =0, k=0.0

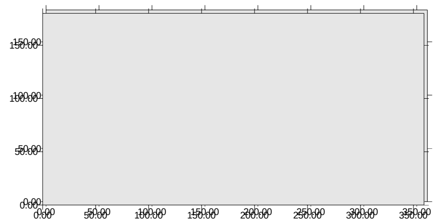
m =0

Analytical modes



Single-layer approach

Double-layer approach



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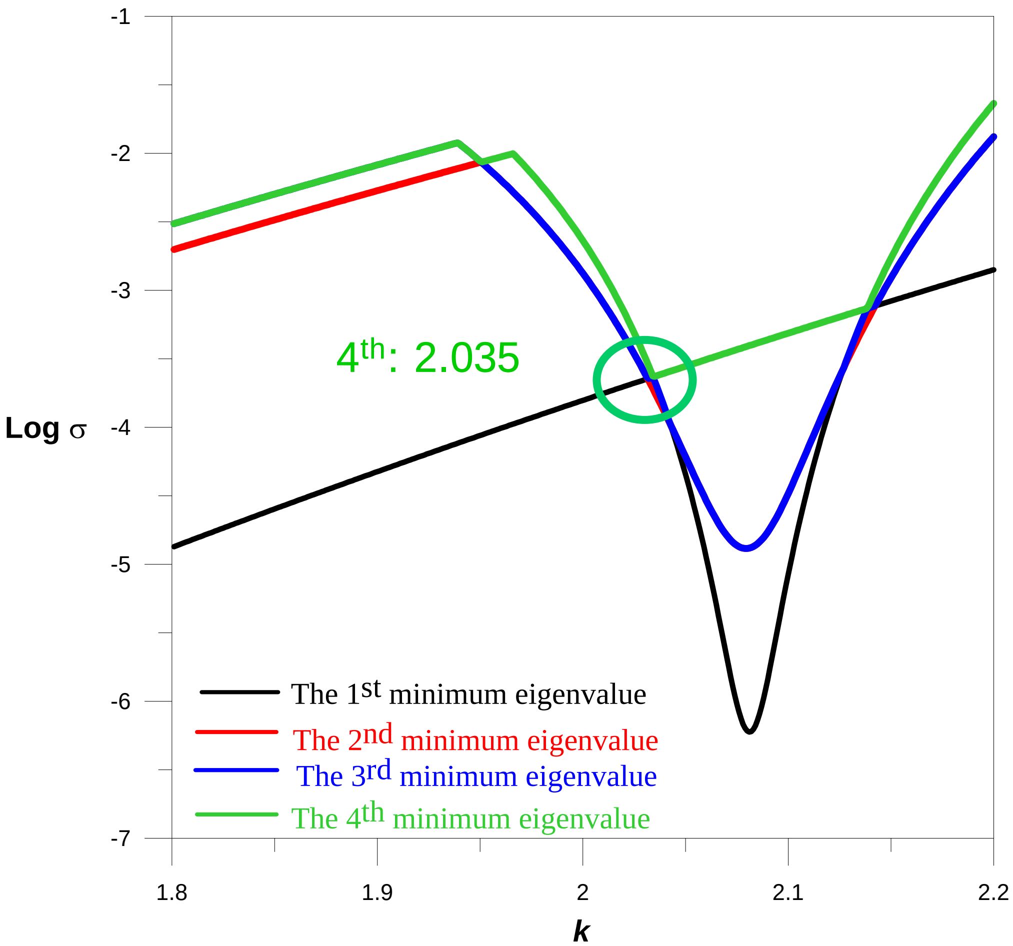
Multiplicity of eigenvalues

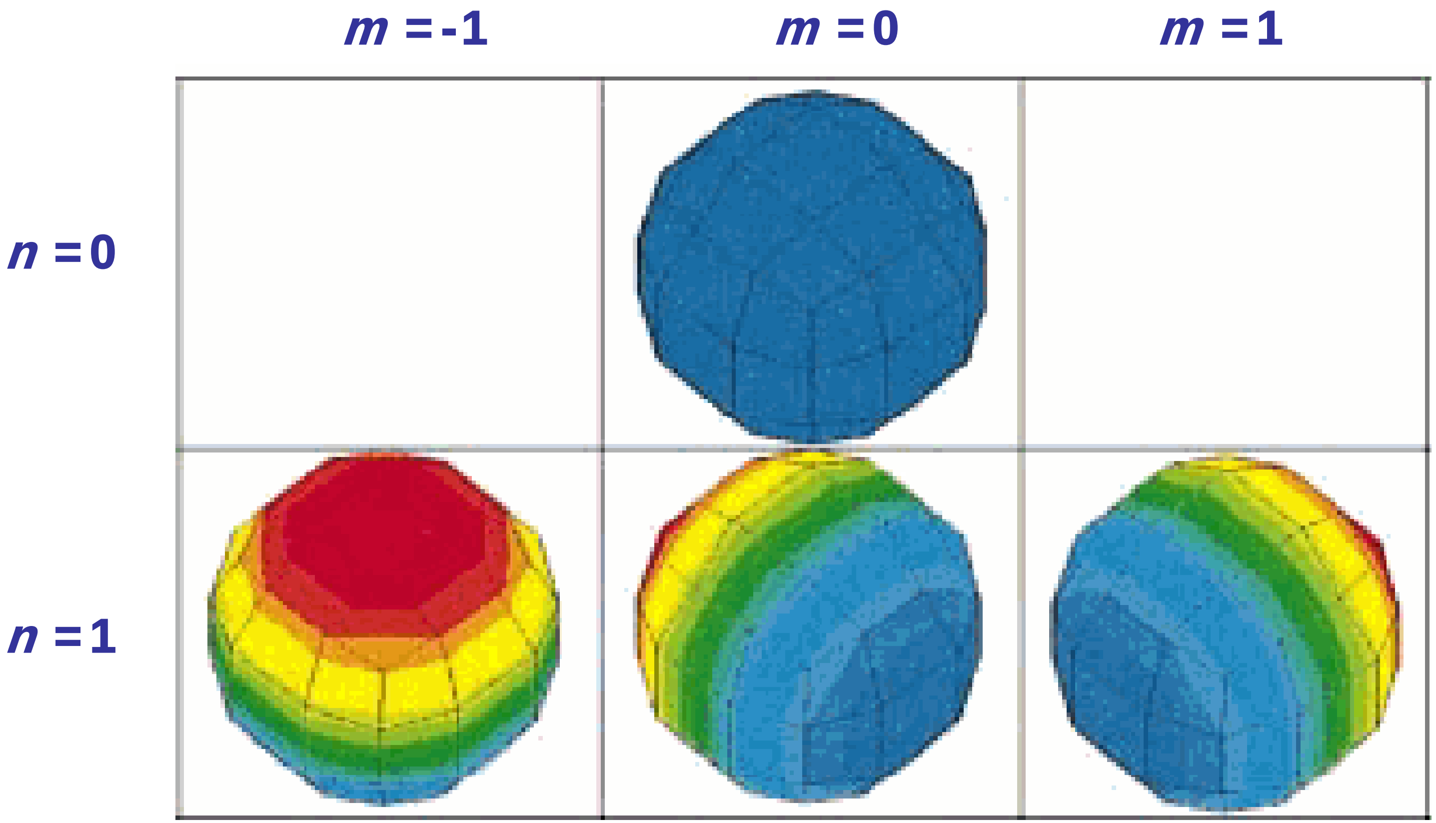


i^{th} minimum eigenvalue	Analytical eigenvalue	Numerical eigenvalue	Test
1st	2.082	2.081	V
2nd		2.079	V
3rd		2.079	V
4th		2.035	ξ

Multiplicity=3

Spherical harmonics

$$u(1,\theta,\phi)=\begin{cases}1, & n=m=0\\ p_n^m(\cos\theta)e^{im\phi}, & \text{otherwise}\end{cases}$$

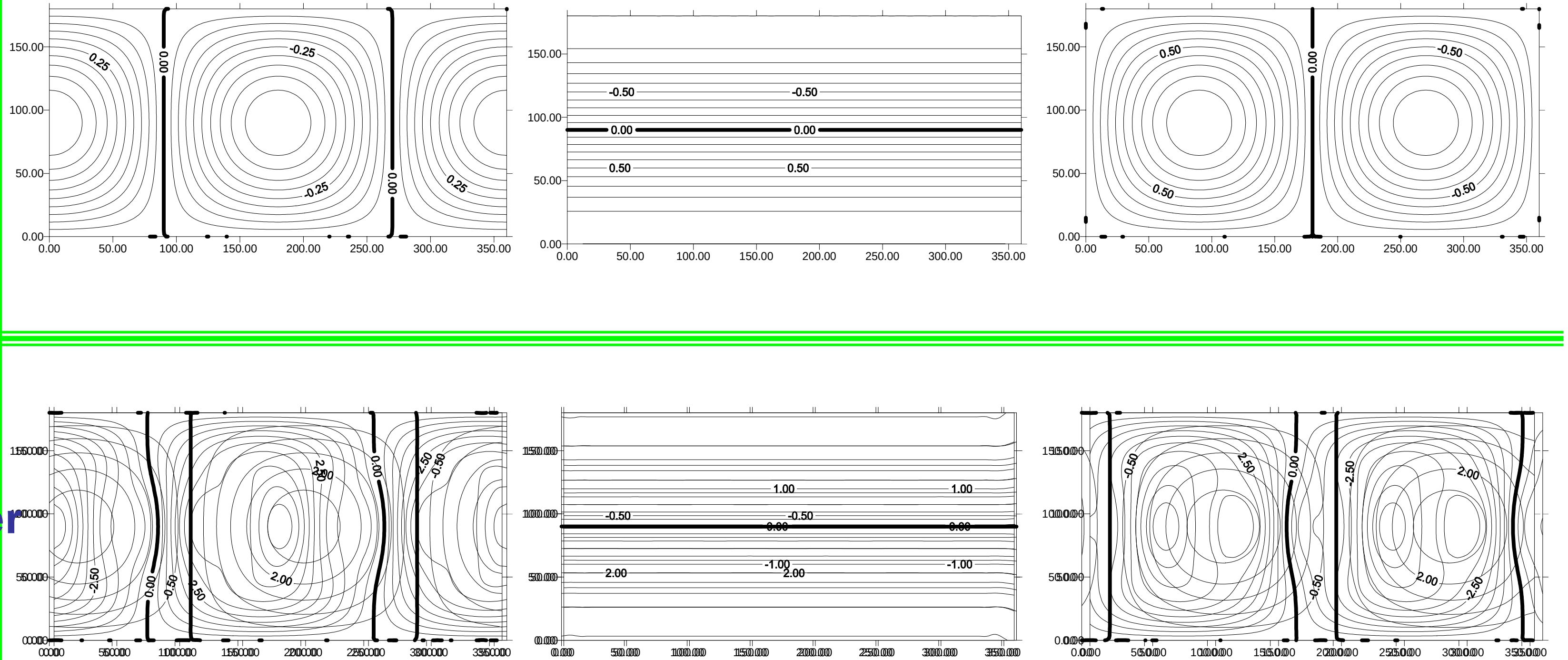


P. A. NELSON
Y. KAHANA

Contours of the boundary modes

n **=1,** ***k*****=2.082**

	m **=-1**	***m*** **=0**	***m*** **=-1**
Analytical modes			
Single-layer approach / **Double-layer approach**			



Outlines

1. Introduction
2. 2-D acoustic eigenproblem
3. SVD updating terms
4. SVD updating documents
5. 3-D cavity
6. Conclusions

Conclusions

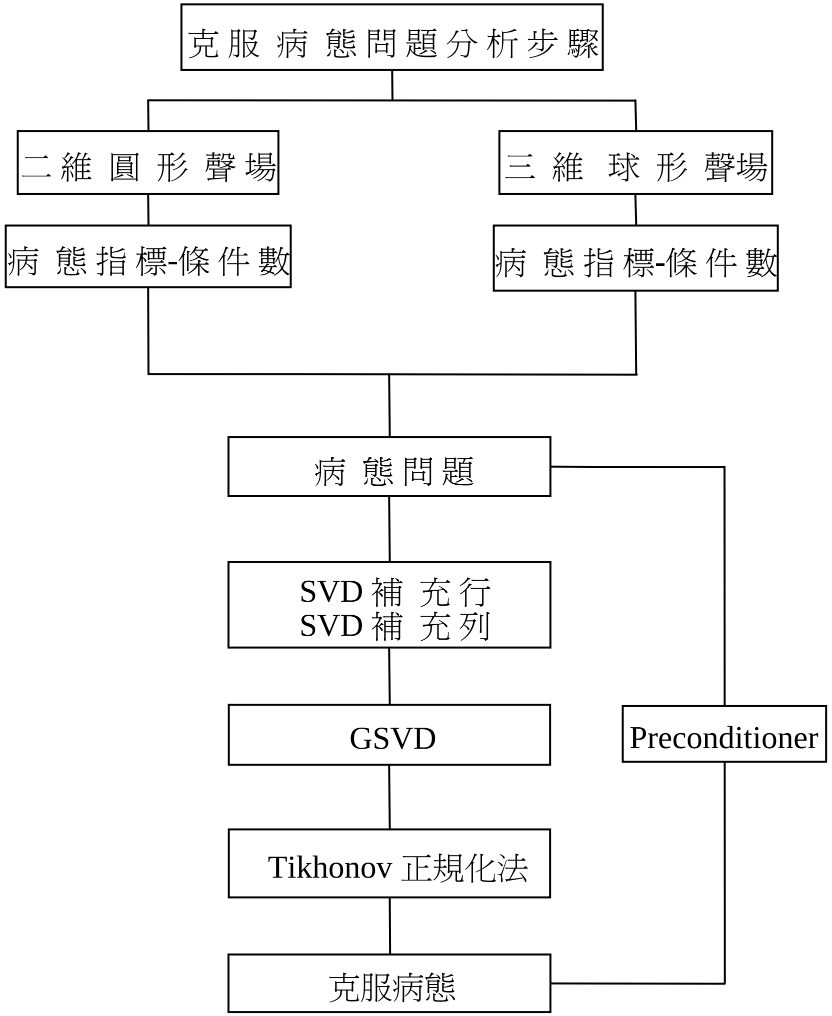
I. Only the boundary nodes are required such that the influence matrices can be determined by the two-point function.

II. The diagonal elements can be derived by invariant method.

III. Based on the imaginary-part formulations, the deficient constraints cause the spurious eigensolutions.

IV. The SVD techniques can solve the true or spurious eigensolutions well.

To overcome the ill-posed problems



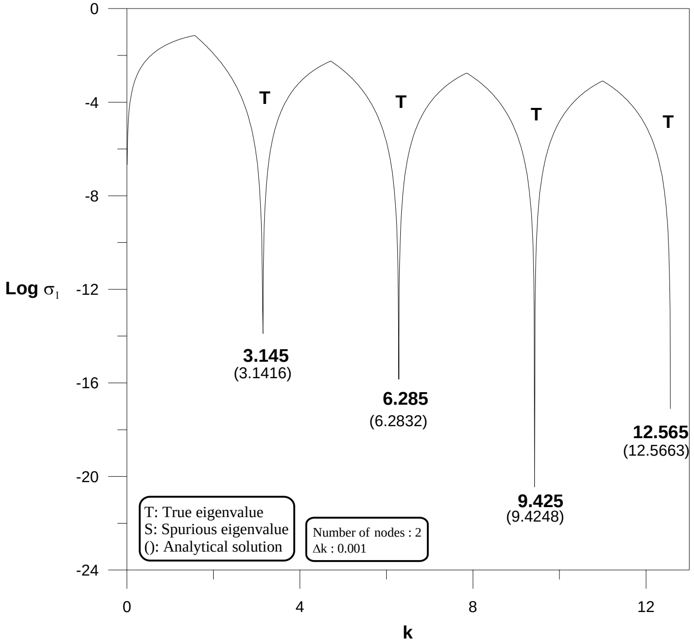
Distributed & concentrated type

	Distributed-type	Concentrated-type*
Single-layer potential approach	Dirichlet problem $u(x)=\int_B U^I(s,x)\phi(s)dB(s)$ Neumann problem $t(x)=\int_B L^I(s,x)\phi(s)dB(s)$	Dirichlet problrm $u(x_i)=\sum_j U^I(s_j,x_i)A_j=(SM)_{ij}A_j$ Neumann problrm $t(x_i)=\sum_j L^I(s_j,x_i)B_j=(SM_x)_{ij}B_j$
Double-layer potential approach	Dirichlet problrm $u(x)=\int_B T^I(s,x)\psi(s)dB(s)$ Neumann problrm $t(x)=\int_B M^I(s,x)\psi(s)dB(s)$	Dirichlet problrm $u(x_i)=\sum_j T^I(s_j,x_i)A_j=(SM_s)_{ij}A_j$ Neumann problrm $t(x_i)=\sum_j M^I(s_j,x_i)B_j=(SM_{sx})_{ij}B_j$

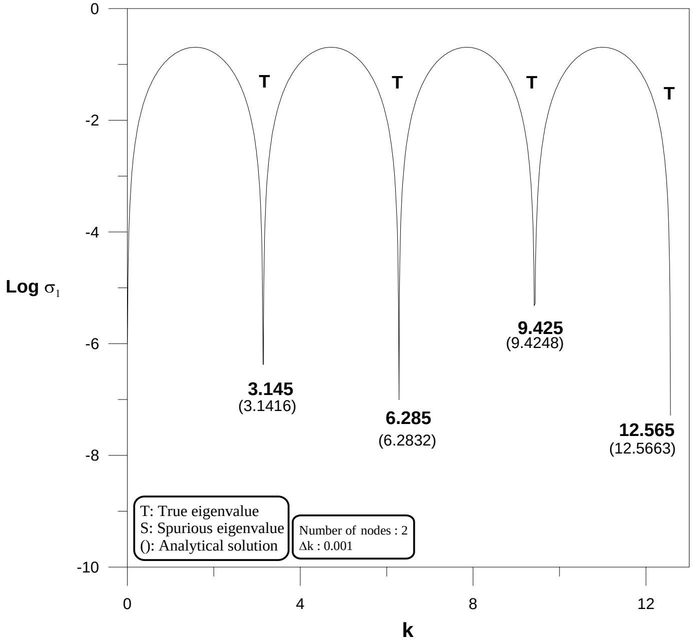
* NDIF method by Kang is the special case

One-dimensional duct --- Dirichlet problem

Single-layer potential approach

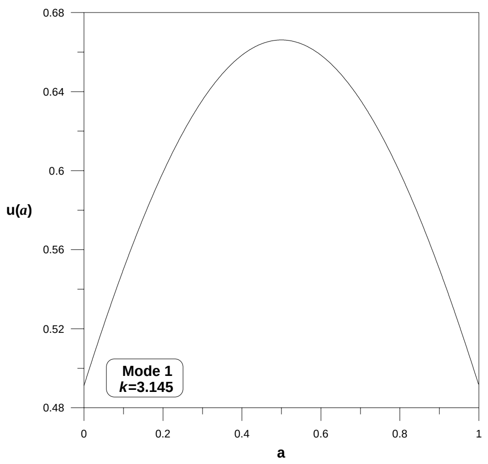


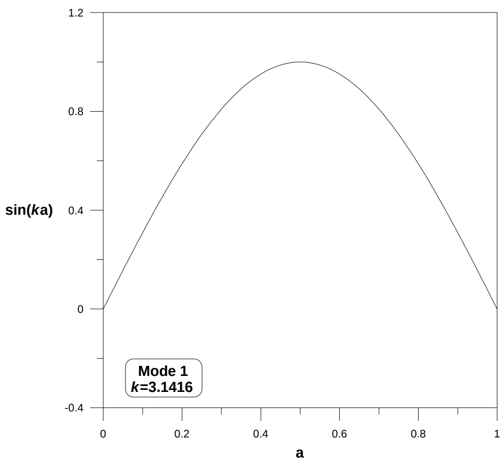
Double-layer potential approach

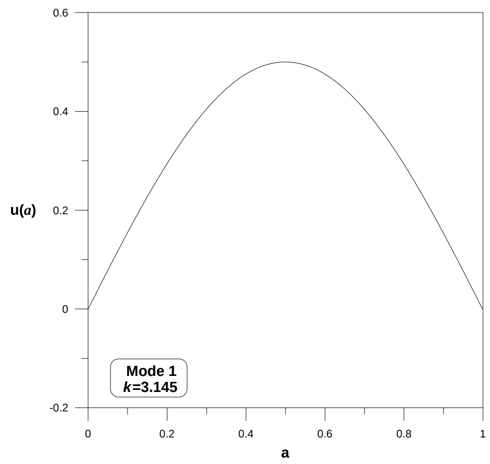


One-dimensional duct --- Dirichlet problem

Single-layer potential approach	Double-layer potential approach	Analytical mode

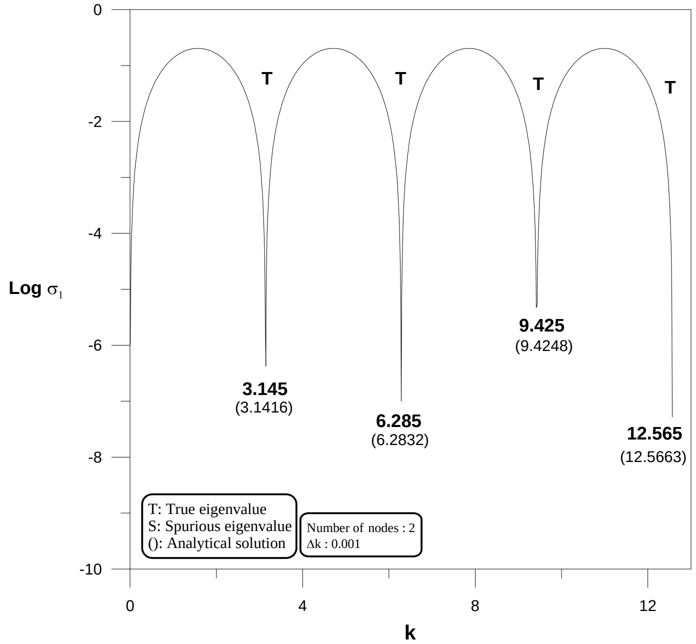




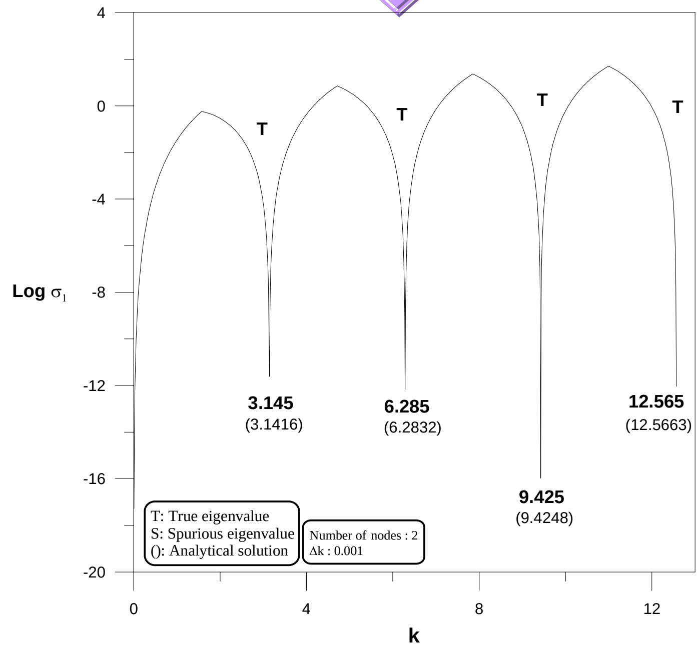


One-dimensional duct --- Neumann problem

Single-layer potential approach



Double-layer potential approach



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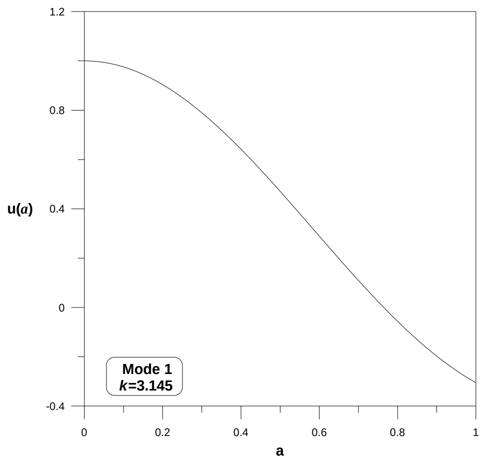
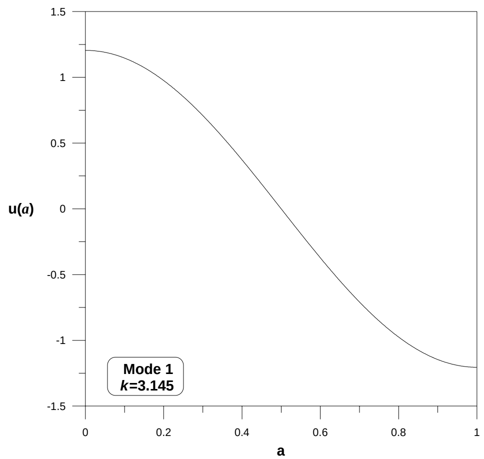
One-dimensional duct --- Neumann problem

Single-layer potential approach



Double-layer potential approach



Analytical mode