工程數學（四）—偏微分方程

I. By using the auxilliary system $U(x,s;t,\tau)$, which is a solution of

$$\frac{\partial^2 U(x,s;t,\tau)}{\partial t^2} - c^2 \frac{\partial^2 U(x,s;t,\tau)}{\partial x^2} = \delta(x-s)\delta(t-\tau), -\infty < x < \infty, t > 0$$

with initial conditions

$$\lim_{t\to\tau} U(x,s;t,\tau) = 0$$

$$\lim_{t\to\tau} \dot{U}(x,s;t,\tau) = 0$$

and no boundary condition since $-\infty < x < \infty$,
Solve the problem

$$u_{tt} = c^2 u_{xx}, \quad for \;\; 0 < x < 1, \;\; t > 0$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = 0$$

and boundary conditions

$$u(0,t) = a(t), u(1,t) = b(t)$$

$$\begin{aligned} \mathbf{u}(\mathbf{x},t) &= \int_0^t \int_B \mathbf{U}(\mathbf{s},\mathbf{x};t,\tau)\cdot\mathbf{t}(\mathbf{s},\tau)dB(\mathbf{s})d\tau - \int_0^t \int_B \mathbf{T}(\mathbf{s},\mathbf{x};t,\tau)\cdot\mathbf{u}(\mathbf{s},\tau)dB(\mathbf{s})d\tau, \\ &= -\int_0^t \mathbf{T}(\mathbf{s},\mathbf{x};t,\tau)\cdot\mathbf{u}(\mathbf{s},\tau)\mid_{s=0}^{s=l} d\tau, \end{aligned} \tag{1}$$

$$\begin{aligned} \mathbf{t}(\mathbf{x},t) &= \int_0^t \int_B \mathbf{L}(\mathbf{s},\mathbf{x};t,\tau)\cdot\mathbf{t}(\mathbf{s},\tau)dB(\mathbf{s})d\tau - \int_0^t \int_B \mathbf{M}(\mathbf{s},\mathbf{x};t,\tau)\cdot\mathbf{u}(\mathbf{s},\tau)dB(\mathbf{s})d\tau \\ &= -\int_0^t \mathbf{M}(\mathbf{s},\mathbf{x};t,\tau)\cdot\mathbf{u}(\mathbf{s},\tau)\mid_{s=0}^{s=l} d\tau, \end{aligned} \tag{2}$$

where $\mathbf{t}(\mathbf{x},t) = \mathbf{u}'(\mathbf{x},t)$ and $\mathbf{U}(\mathbf{s},\mathbf{x};t,\tau), \mathbf{T}(\mathbf{s},\mathbf{x};t,\tau), \mathbf{L}(\mathbf{s},\mathbf{x};t,\tau)$ and $\mathbf{M}(\mathbf{s},\mathbf{x};t,\tau)$ are four kernel functions which can be expressed in terms of series solution

$$\mathbf{U}(\mathbf{s},\mathbf{x};t,\tau) = \sum_{m=1}^{\infty} \frac{1}{N_m\omega_m} sin(\omega_m(t-\tau))\; \mathbf{u}_m(\mathbf{x})\mathbf{u}_m(\mathbf{s}), \tag{3}$$

$$\mathbf{T}(\mathbf{s},\mathbf{x};t,\tau) = \sum_{m=1}^{\infty} \frac{1}{N_m\omega_m} sin(\omega_m(t-\tau))\; \mathbf{u}_m(\mathbf{x})\mathbf{t}_m(\mathbf{s}), \tag{4}$$

$$\mathbf{L}(\mathbf{s},\mathbf{x};t,\tau) = \sum_{m=1}^{\infty} \frac{1}{N_m\omega_m} sin(\omega_m(t-\tau))\; \mathbf{t}_m(\mathbf{x})\mathbf{u}_m(\mathbf{s}), \tag{5}$$

$$\mathbf{M}(\mathbf{s},\mathbf{x};t,\tau) = \sum_{m=1}^{\infty} \frac{1}{N_m\omega_m} sin(\omega_m(t-\tau))\; \mathbf{t}_m(\mathbf{x})\mathbf{t}_m(\mathbf{s}), \tag{6}$$

in which $\mathbf{u}_m(\mathbf{x}) = sin(m\pi x/l)$, $\mathbf{t}_m(\mathbf{x}) = m\pi cos(m\pi x/l)/l$, $N_m = l/2$, $\omega_m = m\pi c/l$, D is $(0,l)$ and B is 0, and l.

$$U(x,s;t,\tau)=0,\ if\ t<\tau$$

$$U(s,x;\tau,t)=0,\ if\ \tau<t$$

$$U(x,s;t,\tau)=U(s,x;t,\tau)$$

$$U(x,s;t,\tau)\neq U(x,s;\tau,t)$$

$$U(x,s;t,\tau)\neq U(s,x;\tau,t)$$

However, both $U(x,s;t,\tau)$ and $U(s,x;\tau,t)$ satisfy the governing equation

$$\frac{\partial^2 U(x,s;t,\tau)}{\partial t^2}-c^2\frac{\partial^2 U(x,s;t,\tau)}{\partial x^2}=\delta(x-s)\delta(t-\tau),-\infty<x<\infty,t>0$$

$$\frac{\partial^2 U(s,x;\tau,t)}{\partial t^2}-c^2\frac{\partial^2 U(s,x;\tau,t)}{\partial x^2}=\delta(x-s)\delta(t-\tau),-\infty<x<\infty,t>0$$

$$\delta(x-s)=\sum_{m=1}^{\infty}\frac{2}{l}sin(m\pi x/l)sin(m\pi s/l)$$

Exact form:

$$U(x,s;t,\tau)=\frac{1}{2c}H(x-s+c(t-\tau))-\frac{1}{2c}H(x-s-c(t-\tau))$$

Series form:

$$\mathbf{U}(\mathbf{s},\mathbf{x};t,\tau)=\sum_{m=1}^{\infty}\frac{1}{N_m\omega_m}sin(\omega_m(t-\tau))\ sin(m\pi x/l)sin(m\pi s/l)$$

Choosing $u=u(x,t)$ and $v(x,t)=U(s,x;\tau,t)$,

$$\begin{aligned}u(s,\tau)&=\int_0^{\tau^+}\int_V u(x,t)(\delta(x-s)\delta(t-\tau))dV(x)dt\\&=\int_V[u(x,t)\dot{U}(s,x;\tau,t)-U(s,x;\tau,t)\dot{u}(x,t)]\ |_{t=0}^{t=\tau^+}\ dV(x)\\&\quad-c^2\int_0^{\tau^+}\int_S[u(x,t)\frac{\partial U(s,x;\tau,t)}{\partial n(x)}-U(s,x;\tau,t)t(x,t)]dS(x)dt\\&=\int_V[U(s,x;\tau,0)\dot{u}(x,0)-u(x,0)\dot{U}(s,x;\tau,0)]dV(x)\\&\quad-c^2\int_0^{\tau^+}\int_S[u(x,t)\frac{\partial U(s,x;\tau,t)}{\partial n(x)}-U(s,x;\tau,t)t(x,t)]dS(x)dt\end{aligned}\qquad(7)$$

After using the casuality conditions

$$U(s,x;\tau,\tau^+)=0$$

$$\dot{U}(s,x;\tau,\tau^+)=0$$

$$
\begin{aligned}
u(x,t) &= \int_0^{\tau^+}\int_V u(s,\tau)(\delta(x-s)\delta(t-\tau))dV(s)d\tau \\
&= \int_V [u(s,\tau)\dot{U}(x,s;t,\tau) - U(x,s;t,\tau)\dot{u}(s,\tau)] \mid_{\tau=0}^{\tau=\tau^+} dV(s) \\
&\quad -c^2\int_0^{\tau^+}\int_S [u(s,\tau)\frac{\partial U(x,s;t,\tau)}{\partial n(s)} - U(x,s;t,\tau)t(s,\tau)]dS(s)d\tau \\
&= \int_V [U(x,s;t,0)\dot{u}(s,0) - u(s,0)\dot{U}(x,s;t,0)]dV(s) \\
&\quad -c^2\int_0^{\tau^+}\int_S [u(s,\tau)\frac{\partial U(x,s;t,\tau)}{\partial n(s)} - U(x,s;t,\tau)t(s,\tau)]dS(s)d\tau
\end{aligned} \tag{8}
$$

Since $U(s,x;t,\tau) = U(x,s;t,\tau)$, we have

$$
\begin{aligned}
u(x,t) &= \int_0^{\tau^+}\int_V u(s,\tau)(\delta(x-s)\delta(t-\tau))dV(s)d\tau \\
&= \int_V [u(s,\tau)\dot{U}(s,x;t,\tau) - U(s,x;t,\tau)\dot{u}(s,\tau)] \mid_{\tau=0}^{\tau=\tau^+} dV(s) \\
&\quad -c^2\int_0^{\tau^+}\int_S [u(s,\tau)\frac{\partial U(s,x;t,\tau)}{\partial n(s)} - U(s,x;t,\tau)t(s,\tau)]dS(s)d\tau \\
&= \int_V [U(s,x;t,0)\dot{u}(s,0) - u(s,0)\dot{U}(s,x;t,0)]dV(s) \\
&\quad -c^2\int_0^{\tau^+}\int_S [u(s,\tau)\frac{\partial U(s,x;t,\tau)}{\partial n(s)} - U(s,x;t,\tau)t(s,\tau)]dS(s)d\tau
\end{aligned} \tag{9}
$$

Derivation of D'Alembert' solution by U, T, L, M kernels.

$$
u(x,t) = \frac{1}{2}\phi(x+ct) + \frac{1}{2}\phi(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds
$$

where

$$
\begin{aligned}
u(x,0) &= \phi(x) \\
\dot{u}(x,0) &= \psi(x)
\end{aligned}
$$

$$
\begin{aligned}
U(s,x;t,\tau) &= U(x,s;t,\tau) \\
T(s,x;t,\tau) &= \frac{\partial U(s,x;t,\tau)}{\partial n(s)} \\
L(s,x;t,\tau) &= \frac{\partial U(s,x;t,\tau)}{\partial n(x)} \\
M(s,x;t,\tau) &= \frac{\partial^2 U(s,x;t,\tau)}{\partial n(x)\partial n(s)}
\end{aligned} \tag{10}
$$