Meshless Method

J. T. Chen

Department of Harbor and River Engineering

National Taiwan Ocean University

May 6, 2003

Topics

- **Part 1:**

  Equivalence of method of fundamental solutions and Trefftz method

- **Part 2:**

  Membrane eigenproblem

- **Part 3:**

  Plate eigenproblem

Part 1

- Description of the Laplace problem
- Trefftz method
- Method of fundamental solutions (MFS)
- Connection between the Trefftz method and the MFS for Laplace equation
- Numerical examples
- Concluding remarks
- Further research

- **Description of the Laplace problem**
- **Trefftz method**
- **Method of fundamental solutions (MFS)**
- **Connection between the Trefftz method and the MFS for Laplace equation**
- **Numerical examples**
- **Concluding remarks**
- **Further research**

Description of the Laplace problem

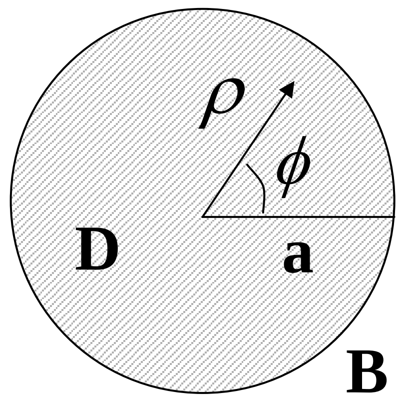
Engineering applications:

1. **Seepage problem**
2. **Heat conduction**
3. **Electrostatics**
4. **Torsion bar**

Two-dimensional Laplace problem with a circular domain

G.E. : $\nabla^2 u(x) = 0, \quad x \in D$

B.C. : $u(x) = \bar{u}, \quad x \in B$



where

∇^2 denotes the Laplacian operator

$u(x)$ is the potential function

ρ is the radius of the field point

ϕ is the angle along the field point

Analytical solution

Field Solution:

$$u(\rho,\phi) = \bar{a}_0 + \sum_{n=1}^{N} \bar{a}_n \left(\frac{\rho}{a}\right)^n \cos(n\phi) + \sum_{n=1}^{N} \bar{b}_n \left(\frac{\rho}{a}\right)^n \sin(n\phi)$$

where $o < \rho < a$

Boundary Condition: **Dirichlet type**

$$u(a,\phi) = \bar{a}_0 + \sum_{n=1}^{N} \bar{a}_n \cos(n\phi) + \sum_{n=1}^{N} \bar{b}_n \sin(n\phi)$$

where $\bar{a}_0, \bar{a}_n, \bar{b}_n$ **are the Fourier coefficients**

- Description of the Laplace problem
- Trefftz method
- Method of fundamental solutions (MFS)
- Connection between the Trefftz method and the MFS for Laplace equation
- Numerical Examples
- Concluding remarks
- Further research

Trefftz method

Representation of the field solution :

$$u(x) = \sum_{j=1}^{2N_T+1} w_j u_j(x)$$

where

$2N_T+1$ **is the number of complete functions**

w_j **is the unknown coefficient**

u_j **is the T-complete function which satisfies the Laplace equation**

T-complete set

T-complete set functions :

$$1,\ \rho^n \cos(n\phi),\ \rho^n \sin(n\phi)$$

⇩

$$u(\rho,\phi) = a_0 + \sum_{n=1}^{N_T} a_n \rho^n \cos(n\phi) + \sum_{n=1}^{N_T} b_n \rho^n \sin(n\phi),\ 0 < \rho < a$$

$$w_j \rightarrow a_0, a_1, b_1, \ldots a_n, b_n \quad n = 0,1,2\ldots.$$

By matching the boundary condition at $\rho = a$

$$u(a,\phi) = a_0 + \sum_{n=1}^{N_T} a_n \rho^n \cos(n\phi) + \sum_{n=1}^{N_T} b_n \rho^n \sin(n\phi).$$

$$u(a,\phi) = \overline{a}_0 + \sum_{n=1}^{N} \overline{a}_n \cos(n\phi) + \sum_{n=1}^{N} \overline{b}_n \sin(n\phi)$$

$$a_0 = \overline{a}_0,$$

$$a_n = \frac{\overline{a}_n}{a^n}, \qquad n = 1,2,\ldots N_T$$

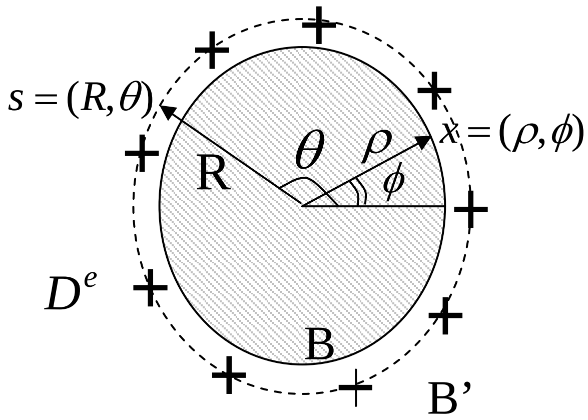
$$b_n = \frac{\overline{b}_n}{a^n} \qquad n = 1,2,\ldots N_T$$

- **Description of the Laplace problem**
- **Trefftz method**
- **Method of fundamental solutions (MFS)**
- **Connection between the Trefftz method and the MFS for Laplace equation**
- **Numerical Examples**
- **Concluding remarks**
- **Further research**

Method of Fundamental Solutions $(R > \rho)$

Field solution :

$$u(x) = \sum_{j=1}^{N_M} c_j U(x, s_j), \quad s_j \in D^e$$



where

N_M **is the number of source points in the MFS**

c_j **is the unknown coefficient**

$U(x, s_j)$ **is the fundamental solution**

D^e **is the complementary domain**

s **is the source point**

x **is the collocation point**

Green's function

$$G(x,s)=\begin{cases}\dfrac{y_1(x)y_2(s)}{W(y_1,y_2)}, & 0\le x\le s\\[2ex] \dfrac{y_1(s)y_2(x)}{W(y_1,y_2)}, & s\le x\le l\end{cases}$$

W **means** ***Wronskin*** **determinant**

Degenerate kernel :

$$U(R,\theta,\rho,\phi)=\begin{cases} U^{i}(R,\theta,\rho,\phi)=\ln(R)-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R})^{m}\cos(m(\theta-\phi)), & R>\rho \\ U^{e}(R,\theta,\rho,\phi)=\ln(\rho)-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{R}{\rho})^{m}\cos(m(\theta-\phi)), & R<\rho \end{cases}$$

Symmetry property for kernel :

$$U(x,s_j)=U(s_j,x) \Longrightarrow u(x)=\sum_{j=1}^{N_M}c_jU(s_j,x),\ \ s_j\in D^{e}.$$

Derivation of degenerate kernel

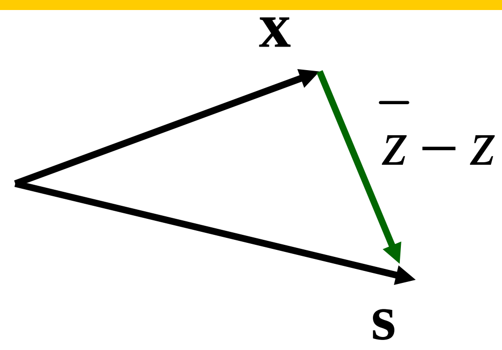
Use the Complex Variable method to derive the degenerate kernel:

Motivation: $z = \ln r + i\theta$

$$\underset{\sim}{x} = (\rho, \phi) \rightarrow \bar{z}, \qquad \underset{\sim}{s} = (R, \theta) \rightarrow z$$

$$\Rightarrow \quad \begin{aligned} \bar{z} &= \ln \rho + i\phi \quad (x) \\ z &= \ln R + i\theta \quad (s) \end{aligned}$$

Derivation of degenerate kernel



Not important

$$\ln(\bar{z}-z)=\ln r+i\theta$$

$$\Rightarrow \quad \ln r=\mathrm{Re}[\ln(\bar{z}-z)]$$

$$\Rightarrow \quad \ln(\bar{z}-z)=\ln \bar{z}(1-\frac{z}{\bar{z}})$$

Due to: $\left|\frac{z}{\bar{z}}\right|<1$ $\quad\Rightarrow\quad \ln|1-x|=-\sum_{m=1}^{\infty}\frac{1}{m}(x)^m$

$$\Rightarrow \quad \mathrm{Re}[\ln \bar{z}+\sum(\frac{-1}{m})(\frac{z}{\bar{z}})^m]=\ln\rho+\sum(\frac{-1}{m})(\frac{R}{\rho})^m\cos(m(\theta-\phi))$$

$$u(\rho,\phi)=\sum_{j=1}^{N_M} c_j[\ln(R)-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R})^m \cos(m(\theta_j-\phi))]$$

By matching the boundary condition $\rho = R = a$

$$u(a,\phi)=\bar{a}_0+\sum_{n=1}^{N}\bar{a}_n\cos(n\phi)+\sum_{n=1}^{N}\bar{b}_n\sin(n\phi)$$

$$\bar{a}_0=\sum_{j=1}^{N_M} c_j\ln(R)$$

$$\frac{\bar{a}_n}{a^n}=-\sum_{j=1}^{N_M} c_j\frac{1}{n}(\frac{1}{R})^n\cos(n\theta_j),\quad n=1,2,\dots N_M$$

$$\frac{\bar{b}_n}{a^n}=-\sum_{j=1}^{N_M} c_j\frac{1}{n}(\frac{1}{R})^n\sin(n\theta_j),\quad n=1,2,\dots N_M$$

- **Description of the Laplace problem**
- **Trefftz method**
- **Method of fundamental solutions (MFS)**
- **Connection between the Trefftz method and the MFS for Laplace equation**
- **Numerical Examples**
- **Concluding remarks**
- **Further research**

On the equivalence of Trefftz method and MFS for Laplace equation

We can find that the T-complete functions of Trefftz method are imbedded in the degenerate kernels of MFS :

MFS:
$$U(x,s)=\begin{cases} U^i(x,s)=\ln R-\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho}{R}\right)^m\cos(m(\theta-\phi)), & \rho<R \\ U^e(x,s)=\ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R}{\rho}\right)^m\cos(m(\theta-\phi)), & \rho>R \end{cases}$$

Trefftz:
$$\rho^m\cos(m\theta),\quad \rho^m\sin(m\theta)$$

By setting $N_T = N_M = 2N+1$

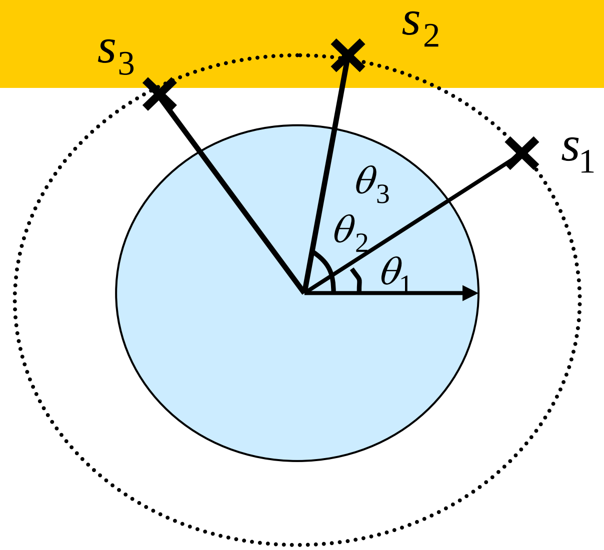
$$a_0 = \sum_{j=1}^{2N+1} c_j \ln(R)$$

$$a_n = -\sum_{j=1}^{2N+1} c_j \frac{1}{n}(\frac{1}{R})^n \cos(n\theta_j), \quad n = 1,2,...2N+1$$

$$b_n = -\sum_{j=1}^{2N+1} c_j \frac{1}{n}(\frac{1}{R})^n \sin(n\theta_j), \quad n = 1,2,...2N+1$$

$$\{\underline{u}\} = [K]\{\underline{v}\}$$

Trefftz **MFS**



in which

$$\langle w_1\rangle = \ln(R)[1, 1.......,1],$$

$$\langle w_2\rangle = (\frac{-1}{R})[\cos(\theta_1), \cos(\theta_2)..., \cos(\theta_{2N+1})],$$

$$\langle w_3\rangle = (\frac{-1}{R})[\sin(\theta_1), \sin(\theta_2)..., \sin(\theta_{2N+1})],$$

$$\vdots$$

$$\langle w_{2N}\rangle = \frac{-1}{n}(\frac{1}{R})^n[\cos(N\theta_1), \cos(N\theta_2)..., \cos(N\theta_{2N+1})],$$

$$\langle w_{2N+1}\rangle = \frac{-1}{n}(\frac{1}{R})^n[\sin(N\theta_1), \sin(N\theta_2)..., \sin(N\theta_{2N+1})],$$

$$K = \begin{bmatrix} \langle w_1\rangle \\ \langle w_2\rangle \\ \vdots \\ \vdots \\ \langle w_{2N+1}\rangle \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$[K] = [T_R][T_\theta]$$

$$[T_\theta] = \begin{bmatrix} 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cos(\theta_1) & \cos(\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(\theta_{2N+1}) \\ \sin(\theta_1) & \sin(\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(\theta_{2N+1}) \\ \cos(2\theta_1) & \cos(2\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(2\theta_{2N+1}) \\ \sin(2\theta_1) & \sin(2\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(2\theta_{2N+1}) \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cos(N\theta_1) & \cos(N\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(N\theta_{2N+1}) \\ \sin(N\theta_1) & \sin(N\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(N\theta_{2N+1}) \end{bmatrix}_{(2N+1)\times(2N+1)}$$

Matrix T_θ

$$[T_\theta]=\begin{bmatrix} 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cos(\theta_1) & \cos(\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(\theta_{2N+1}) \\ \sin(\theta_1) & \sin(\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(\theta_{2N+1}) \\ \cos(2\theta_1) & \cos(2\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(2\theta_{2N+1}) \\ \sin(2\theta_1) & \sin(2\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(2\theta_{2N+1}) \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cos(N\theta_1) & \cos(N\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(N\theta_{2N+1}) \\ \sin(N\theta_1) & \sin(N\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(N\theta_{2N+1}) \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$[T_\theta][T_\theta]^T=\begin{bmatrix} 2N+1 & 0 & \cdots & \cdots & 0 \\ 0 & \frac{2N+1}{2} & \cdots & \cdots & 0 \\ 0 & 0 & \frac{2N+1}{2} & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{2N+1}{2} \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$$\det[T_\theta]=\frac{(2N+1)^{N+\frac{1}{2}}}{2^N}\neq 0, \quad N\in Natural\ number$$

Matrix T_R

$$[T_R]=\begin{bmatrix} \ln(R) & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \frac{-1}{R} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \frac{-1}{R} & 0 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \frac{-1}{2}(\frac{1}{R})^2 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \frac{-1}{2}(\frac{1}{R})^2 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \frac{-1}{N}(\frac{1}{R})^N & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{-1}{N}(\frac{1}{R})^N \end{bmatrix}_{(2N+1)\times(2N+1)}$$

$R >> a$

$N \to \infty$

ill-posed problem

$\ln 1 = 0$

$(R = 1)$

Degenerate scale problem

Papers of degenerate scale (Taiwan)

1. J. T. Chen, S. R. Kuo and J. H. Lin, 2002, Analytical study and numerical experiments for degenerate scale problems in the boundary element method for two-dimensional elasticity, *Int. J. Numer. Meth. Engng.*, Vol.54, No.12, pp.1669-1681. (SCI and EI)
2. J. T. Chen, C. F. Lee, I. L. Chen and J. H. Lin, 2002 An alternative method for degenerate scale problem in boundary element methods for the two-dimensional Laplace equation, *Engineering Analysis with Boundary Elements*, Vol.26, No.7, pp.559-569. (SCI and EI)
3. J. T. Chen, J. H. Lin, S. R. Kuo and Y. P. Chiu, 2001, Analytical study and numerical experiments for degenerate scale problems in boundary element method using degenerate kernels and circulants, *Engineering Analysis with Boundary Elements*, Vol.25, No.9, pp.819-828. (SCI and EI)
4. J. T. Chen, S. R. Lin and K. H. Chen, 2003, Degenerate scale for Laplace equation using the dual BEM, *Int. J. Numer. Meth. Engng, Revised.*

Papers of degenerate scale (China)

1. 胡海昌, 平面調和函數的充要的邊界積分方程, 1992, *中國科學學報*, Vol.4, pp.398-404
2. 胡海昌, 調和函數邊界積分方程的充要條件, 1989, *固體力學學報*, Vol.2, No.2, pp.99-104
3. W. J. He, H. J. Ding and H. C. Hu, Nonuniqueness of the conventional boundary integral formulation and its elimination for two-dimensional mixed potential problems, Computers and Structures, Vol.60, No.6, pp.1029-1035, 1996.

The efficiency between the Trefftz method and the MFS

We propose an example for exact solution:

$$u(r,\theta) = r^{50}\cos(50\theta),$$

⇩

Trefftz method :

N_T=50

⇩

N=101 terms

MFS :

N_M<50

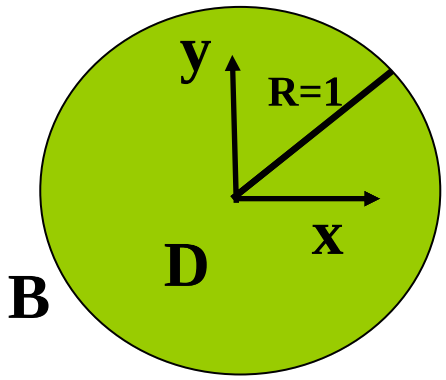
⇩

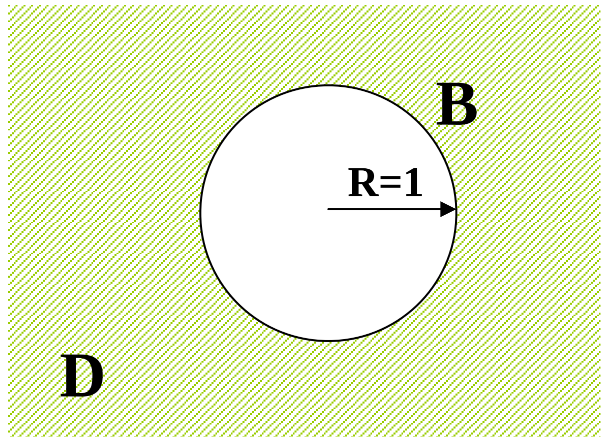
N < 101 terms

Numerical Examples

$$G.E.: \Delta u(x) = 0$$

$$B.C.: u(x) = \cos(3\theta)$$





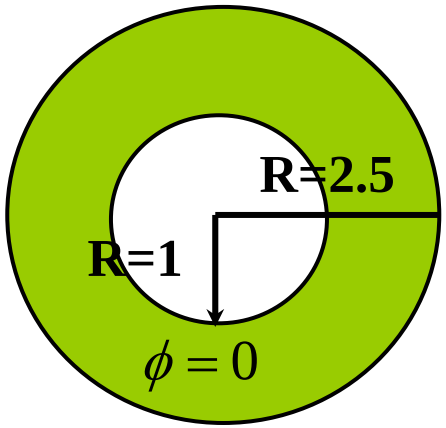
Exact solution: $u(r,\theta) = r^3 \cos(3\theta)$ $\quad u(r,\theta) = c \ln r + \frac{1}{r^3}\cos(3\theta)$

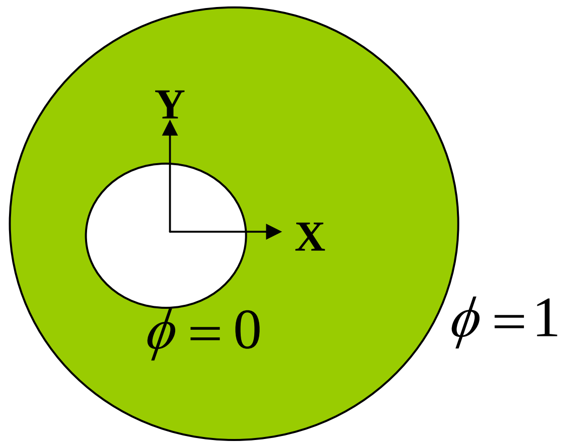
1. **Trefftz method for simply-connected problem**
2. **MFS for simply-connected problem**

Numerical Examples

$G.E.: \Delta\phi = 0$

小圓半徑為1;大圓半徑為2.5



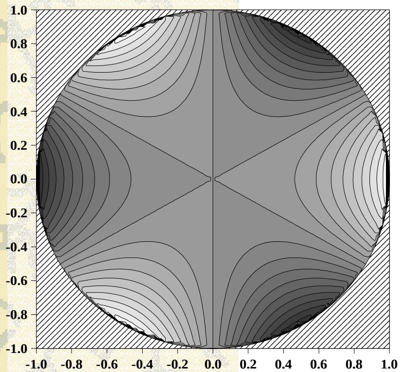
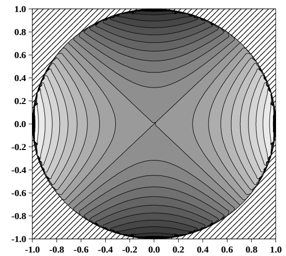
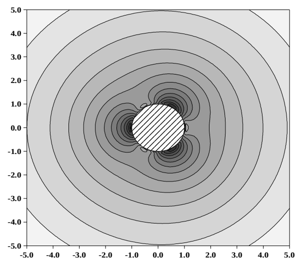
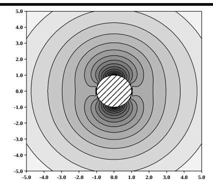
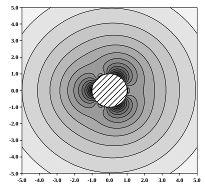
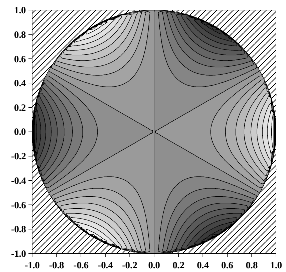
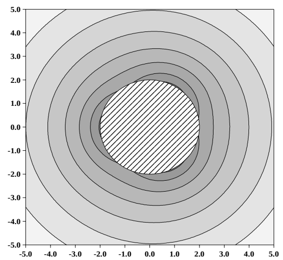
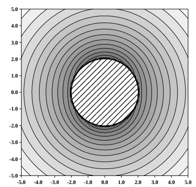
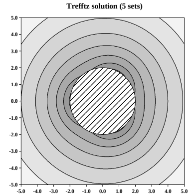


Exact solution: $u(\rho,\phi) = \frac{\ln\rho}{\ln 2.5}$ $\quad u(\rho,\phi) = \frac{1}{2\ln 2}\{\frac{16\rho^2+1+8\rho\cos\phi}{\rho^2+16+8\rho\cos\phi}\}$

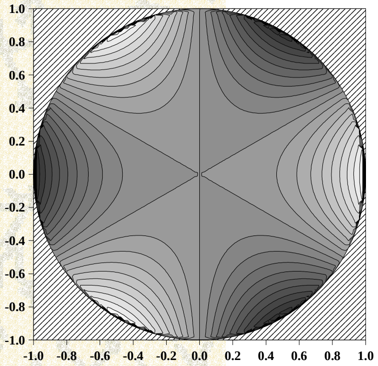
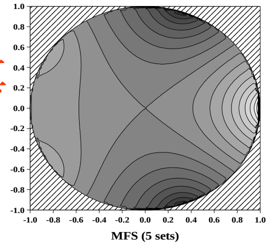
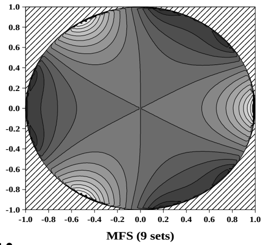
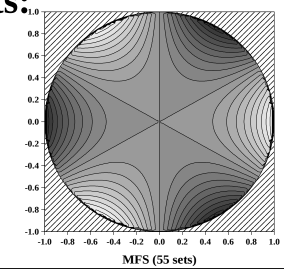
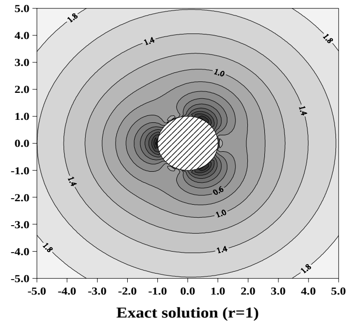
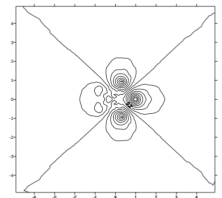
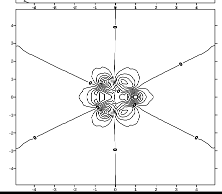
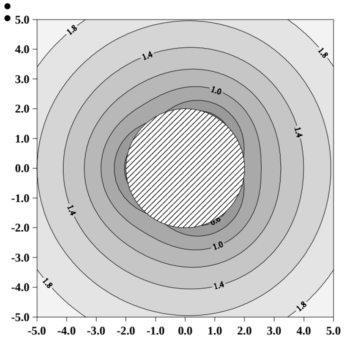
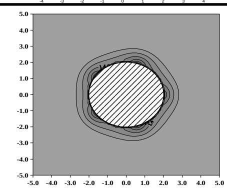
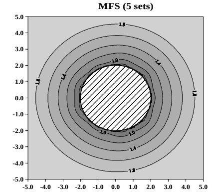
1. **Trefftz method for multiply-connected problem**
2. **MFS for multiply-connected problem**

Numerical Example 1

Trefftz method for simply-connected problem

Interior problem		Exterior problem		
Exact solution	**Numerical solution**	**Exact solution**	**Numerical solution**	
 Exact solution	5 Points: B.C.失真 (基底缺損)  Trefftz method (5 sets)	a=1  Exact solution (r=1)	5 Points:  Trefftz solution (5 sets)	9 Points:  Trefftz solution (9 sets)
	9 Points:  Trefftz method (9 sets)	a=2  Exact solution (r=2)	5 Points:  Trefftz solution (5 sets)	9 Points:  Trefftz solution (9 sets)

Numerical Example 2

<table>
<tr><th colspan="4">MFS for simply-connected problem</th></tr>
<tr><th colspan="2">Interior problem</th><th colspan="2">Exterior problem</th></tr>
<tr><th>Exact solution</th><th>Numerical solution</th><th>Exact solution</th><th>Numerical solution</th></tr>
<tr>
<td rowspan="2">
Exact solution</td>
<td rowspan="2">5 Points:
B.C.失真

MFS (5 sets)
9 Points:

MFS (9 sets)
55 Points:

MFS (55 sets)</td>
<td>a=1:

Exact solution (r=1)</td>
<td>5 Points:

9 Points:
</td>
</tr>
<tr>
<td>a=2:

Exact solution (r=2)</td>
<td>5 Points:
B.C.失真

MFS (5 sets)
9 Points:

MFS (9 sets)</td>
</tr>
</table>

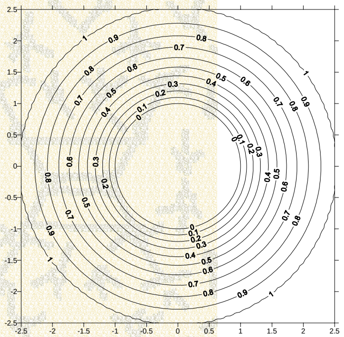
Numerical Example 3

Trefftz method for multiply-connected problem

Concentric circle (以下為等角度分佈)

Exact solution

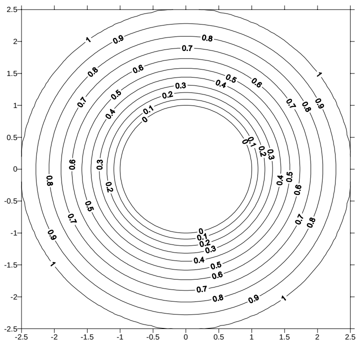
26 Points



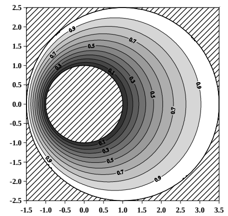
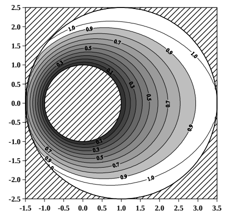
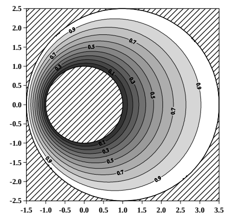
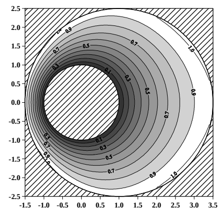
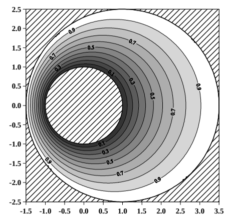
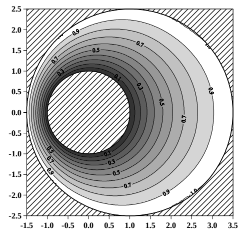
$$u(\rho,\phi)=\frac{\ln\rho}{\ln 2.5}$$

Numerical solution

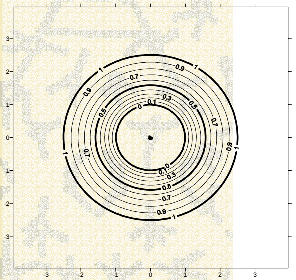
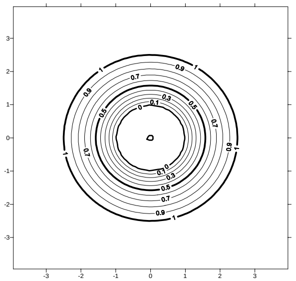
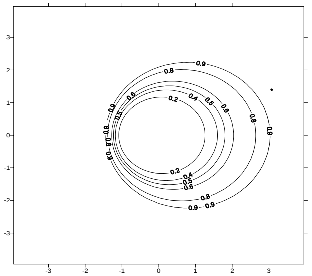
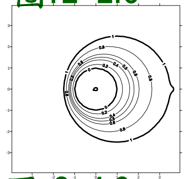
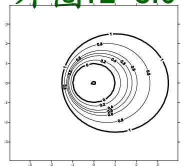
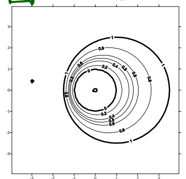
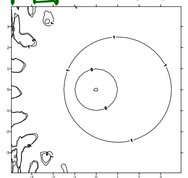
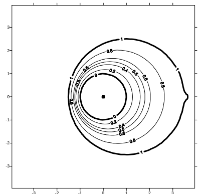
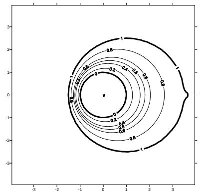
26 Points



Eccentric circle (以下為等角度分佈)

Exact solution		Numerical solution
	**6 Points**	
	**14 Points**	
	**26 Points**	

Numerical Example 4

MFS for multiply-connected problem			
Concentric circle		**Eccentric circle**	
Exact solution	**Numerical solution**	**Exact solution**	**Numerical solution**
內圈佈20點; 外圈60點 $u(\rho,\phi)=\dfrac{\ln\rho}{\ln 2.5}$ 	內圈r=0.9; 外圈r=2.6 內圈佈20點; 外圈60點 	內圈佈20點; 外圈60點 $u(\rho,\phi)=\dfrac{1}{2\ln 2}\times\{\dfrac{16\rho^2+1+8\rho\cos\alpha}{\rho^2+16+8\rho\cos\alpha}\}$ 	內圈佈20點; 外圈60點;內圈r1=0.9 外圈r2=2.6  外圈r2=3.0  外圈r2=4.0  外圈r2=10.0  內圈佈20點; 外圈60點;外圈r2=2.6 內圈r1=0.5  內圈r1=0.3 

- Description of the Laplace problem
- Trefftz method
- Method of fundamental solutions (MFS)
- Connection between the Trefftz method and the MFS for Laplace equation
- Numerical Examples
- Concluding remarks
- Further research

Concluding Remarks

1. **The proof of the mathematical equivalence between the Trefftz method and MFS for Laplace equation was derived successfully.**
2. **The T-complete set functions in the Trefftz method for interior and exterior problems are imbedded in the degenerate kernels of the fundamental solutions as shown in Table 1 for 1-D, 2-D and 3-D Laplace problems.**
3. **The sources of degenerate scale and ill-posed behavior in the MFS are easily found in the present formulation.**
4. **It is found that MFS can approach the exact solution more efficiently than the Trefftz method under the same number of degrees of freedom.**

Comparison between the Trefftz method and MFS

	Trefftz method	MFS
Objectivity (Frame of indifference)	Bad	Good
Degenerate scale	Disappear	Appear
Ill-posed behavior	Appear	Appear

- Description of the Laplace problem
- Trefftz method
- Method of fundamental solutions (MFS)
- Connection between the Trefftz method and the MFS for Laplace equation
- Numerical Examples
- Concluding remarks
- Further research

Further research

Presented

Laplace problem (interior)	→	**Laplace problem (exterior)**
↓		↓
Helmholtz problem (interior)	→	**Helmholtz problem (exterior)**

Simply-connected ⟶ **Multiply-connected ?**

Numerical examples ?

The End

Thanks for your kind attention

Basis of the Laplace equation for Trefftz method

1 → $\nabla^2 1 = 0$

$\rho^n \cos(n\phi)$ → $\nabla^2 \rho^n \cos(n\phi) = 0$

$\rho^n \sin(n\phi)$ → $\nabla^2 \rho^n \sin(n\phi) = 0$

Laplace Equation

where

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}$$

Fundamental solution

$$\nabla_x^2 U(x,s) = \delta(x-s)$$

⇩

$$U(x,s) = \ln(r) \qquad r = |\underline{x} - \underline{s}|$$

where

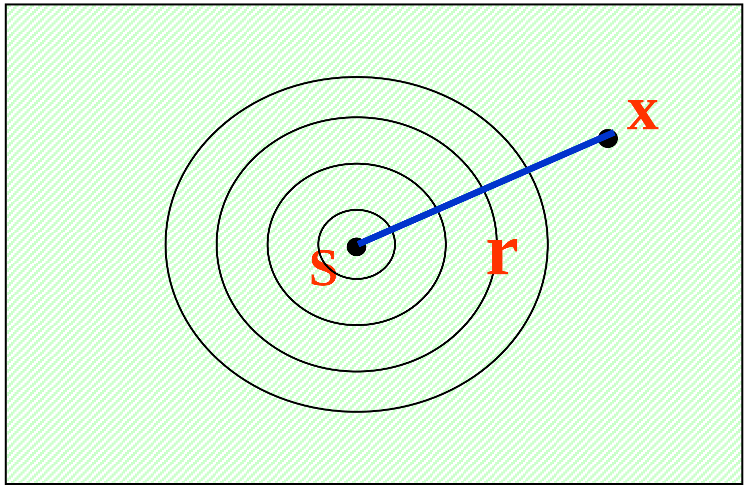
$$\nabla_x^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}$$

$$x = (\rho, \phi)$$

Degenerate kernel (step1)

Step 1

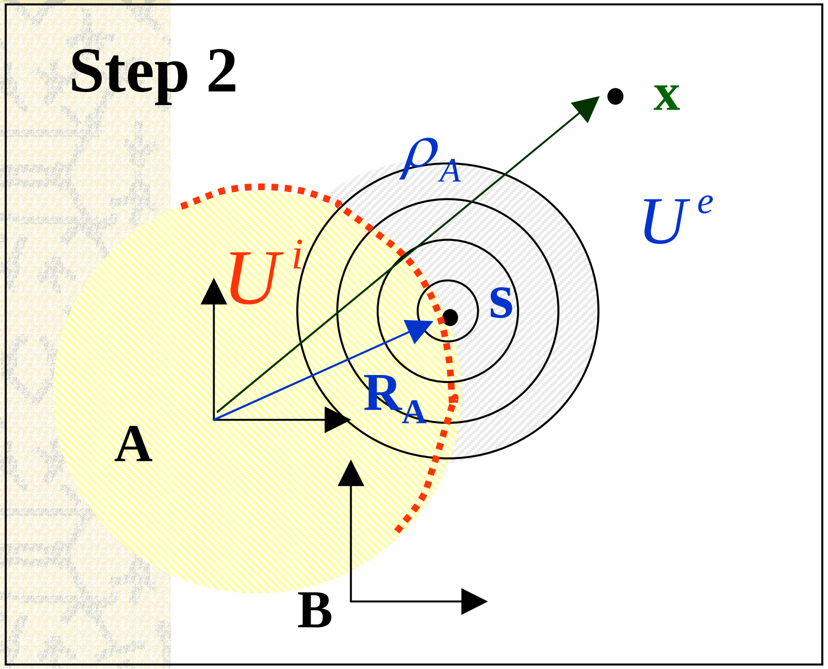
$$U(s, x) = \ln(r) = \ln|\underline{s} - \underline{x}|$$

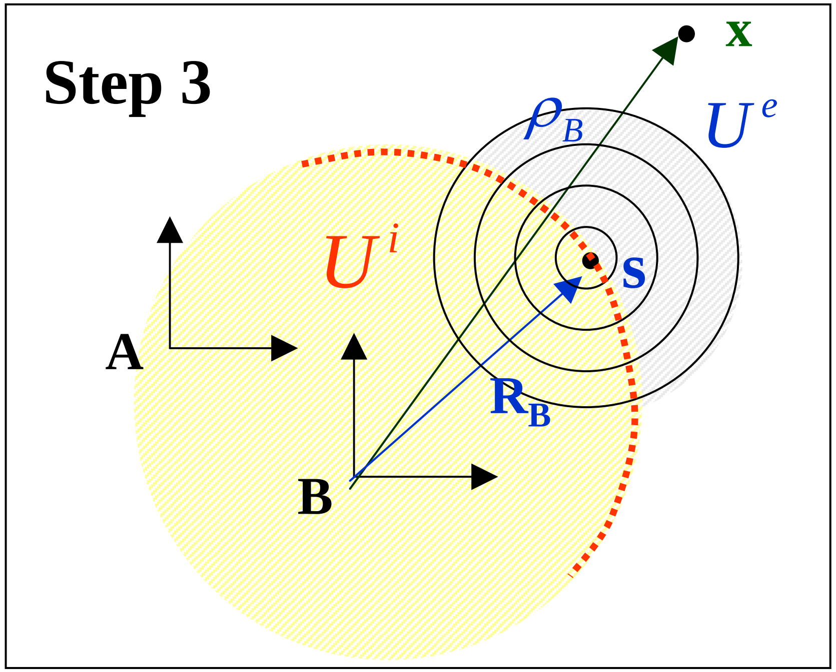


x: variable

s: fixed

Degenerate kernel (Step 2, Step 3)





$$U^i(R,\theta,\rho,\phi) = \ln(\rho) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R}{\rho}\right)^m \cos(m(\theta-\phi)), \quad R > \rho$$

$$U^e(R,\theta,\rho,\phi) = \ln(R) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho}{R}\right)^m \cos(m(\theta-\phi)), \quad R < \rho$$