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LAYER POTENTIALS AND ACOUSTIC DIFFRACTION

P. J. T. Filippi

Laboratoire de Mécanique et d'Acoustique, Centre National de la Recherche Scientifique, 13274 Marseille Cédex 2, France

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Various more or less classical layer potential representations of the diffracted acoustic field within an enclosure or external to an obstacle are discussed. It is shown that it is always possible to find such a representation which yields to an integral equation equivalent to the partial derivative boundary value problem: that is, the conditions of existence and uniqueness of the solution are the same in both formulations. Several numerical experiments are reported, which show that simple and reasonably inexpensive techniques provide predictions of the acoustic field, or of the eigenfrequencies, with an accuracy sufficient for acoustical engineering purposes.

1. INTRODUCTION

The aim of this paper is twofold. First, the integral equations of linear acoustics are presented in a more rigorous mathematical way than that classically used in theoretical acoustics; in particular, the basic concepts of distribution theory are used. Indeed, the best modern mathematical tools for discussing integral equations (existence of solutions, uniqueness, field integral representations, regularity properties, numerical approximations) are those of the recently developed theories of "pseudo-differential operators" [1, 2] and "Poisson pseudo-kernels" [3, 4]. The studies in which these theories have been developed are described by Mikhlin [5] in his book *Multidimensional Singular Integral Equations*; the modern symbolic calculus now developed is of easier and more general use. A significant result (not established here) deals with the so-called "edge conditions" for the diffraction by a thin screen: it can be proved that if a thin screen is considered as the limit of a thick obstacle, the thickness of which is decreasing to zero—and it seems that this is the only way for defining mathematically a thin physical screen—no "edge conditions" are to be imposed, but edge properties can be proved. Another significant result concerns the definition and the numerical approximation of the value on a surface of the normal derivative of a double layer potential, the layer being supported by this surface: the theory of Poisson pseudo-kernels proves that such a quantity is perfectly defined by a limiting process, and justifies the numerical approximation given in section 3.

The second aim of this paper is to gather and explicitly describe numerical experiments which have been proved to be efficient for solving the various problems which can be encountered by the physicist; accordingly treatments are presented of interior and exterior problems, eigenvalue problems and problems of diffraction by thin obstacles. The literature on these problems is plentiful, but the more efficient and general methods for solving difficult cases such as exterior problems, or thin screen diffraction problems, seem to have been often ignored, despite their simplicity.

The remainder of this introductory section is devoted to recalling how a boundary value problem can be replaced by a system of integral equations, and to pointing out some difficulties which appear.

1.1. BOUNDARY VALUE PROBLEMS AND INTEGRAL REPRESENTATION OF THE SOLUTION

Let Ω be a bounded or unbounded domain of space $\mathbb{R}^n$ (for acoustics, $n = 2$ or 3); Γ is the boundary of Ω with normal n pointing out of Ω. One seeks the solution $u(M)$ of a partial differential equation

$$\mathscr{L}u(M) = f(M),$$

where $\mathscr{L}$ is an elliptic partial differential operator of order $2m$ with indefinitely differentiable coefficients (C^∞ coefficients) and f is a function (or more generally a distribution) compactly supported in Ω (i.e., f is zero outside a bounded domain contained in Ω).

The function $u(M)$ and its successive normal derivatives $\partial_n^{(j)}u(M)$ up to order $(2m-1)$ are assigned to satisfy m differential relationships with C^∞ coefficients on Γ (boundary conditions) which can be formally written as

$$l_i(u, \partial_n u, \ldots, \partial_n^{(2m-1)}u)(P) = 0, \qquad P \in \Gamma, i = 1, 2, \ldots, m.$$

The first difficulty which appears is to define the value on Γ of a function like $\partial_n^{(j)}u(M)$ which is perfectly defined when M is closed to a point P of Γ, but remains in Ω; for example:

$$\partial_{n(P)}u(M) = \mathbf{n}(P).\mathbf{grad}_M u(M),$$

$$\partial_{n(P)}^2 u(M) = \mathbf{n}(P).\mathbf{grad}_M[\mathbf{n}(P).\mathbf{grad}_M u(M)],$$

where P is a parameter and the differentiations are taken with respect to M. If u is a function defined and derivable in the whole space, the value on Γ of the preceding quantities is obvious, but when this is not the case some limiting procedure is needed.

If Ω is unbounded, the uniqueness of u is ensured by a Sommerfeld condition at infinity (or any equivalent condition) which expresses the conservation of energy principle from which the governing equation is generally derived.

Define now the elementary kernel $G_S(M)$ which represents the free field response to a spherical point source $\delta_S(M)$ located at S. Let $\mu^j(P)$ be the density of multipole sources of order j supported by Γ. An integral representation of $u(M)$ is given by

$$u(M) = \int_\Omega f(Q)\, G_Q(M)\, \mathrm{d}Q + \sum_{j=0}^{q} (-1)^j \int_\Gamma \mu^j(Q)\, \partial_{n(Q)}^j G_Q(M)\, \mathrm{d}Q,$$

or, by using the notation of distribution theory, u can be written as

$$u(M) = \langle G_Q(M), f(Q)\rangle + \langle G_Q(M), \sum_{j=0}^{q} \left(\frac{\mathrm{d}}{\mathrm{d}n}\right)^j (\mu^j \delta_\Gamma)(Q)\rangle,$$

where $\langle\,,\rangle$ means an integration over Q and δ_Γ is the Dirac measure with support Γ. This representation depends on q functions, the $\mu^j(Q)$, which have to be determined by the m boundary conditions; consequently q is at least equal to m.

1.2. INTEGRAL EQUATIONS AND REGULARITY PROPERTIES OF THEIR SOLUTIONS AND OF u

Introducing the integral representation of $u(M)$ into the boundary conditions, one gets a system of m integral equations and $q+1$ unknowns; if $q+1 > m$, $q+1-m$ additional relationships between the μ^j are necessary.

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2. INTEGRAL EQUATION

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The difficulty arising at this step is to define correctly the values on Γ of $u(M)$ and its successive normal derivatives. It is well known that if the expression of $u(M)$ is written for a point P on Γ, it does not generally represent the value of u on Γ: that is,

$$\int_{\Omega} f(Q)\, G_Q(P)\, \mathrm{d}Q + \sum_{j=0}^{q} (-1)^j \int_{\Gamma} \mu^j(Q)\, \partial^j_{n(Q)}\, G(P, Q)\, \mathrm{d}Q \neq u(P), \qquad P \in \Gamma,$$

because of the discontinuity properties of layer potentials and of the fact that non-integrable kernels can occur.

The theory of Poisson pseudo-kernels ($G_S(M)$ is such a kernel) provides a limiting procedure for defining the value on Γ of the $\partial^j_n u$. The expressions so obtained represent the physical quantities: the jumps of the discontinuous layer potentials are taken into account; the integrals with non-integrable kernels become meaningful by taking the integration in a more general sense than the Riemann one (finite parts of integrals are involved [6]). Though the theory is somewhat complicated, the results are very easy to use, as will be seen later in the discussion of diffraction problems in exterior domains, such as, for example, diffraction by a thin reflecting screen. For this problem it is well known that the integral equation obtained has a non-integrable kernel, but the numerical computation can be performed in a simple way by applying the fact that the integral represents the normal derivative of a layer potential. Thus, the more or less complicated ways generally proposed in the literature to obtain an integrable kernel are unnecessary (for example, many authors suggest a regularization method which leads to an integral equation the kernel of which is an integral over Γ; such a procedure results in uselessly heavy numerical methods).

The system of integral equations determining the μ^j can be written as

$$l_i\left\{\lim_{M\in\Omega\to P\in\Gamma} \partial^k_{n(P)}\left\langle G_Q(M), \sum_{j=0}^{q}\left(\frac{\mathrm{d}}{\mathrm{d}n}\right)^j (\mu^j\delta_\Gamma)(Q)\right\rangle\right\} = -l_i\{\partial^k_{n(P)}\langle G_Q(P), f(Q)\rangle\}, \qquad i = 1, 2, \ldots, m,$$

and $q+1-m$ additional relationships if $q+1-m>0$.

Classically, use is made of the Fredholm alternative; but one has to keep in mind that this theorem has been proved only for regular kernels. Nevertheless, Mikhlin [5] showed that a regularizing procedure (always very complicated, and to be determined for each particular case) enables one to transform the integral system into a Fredholm one. The modern theories, using a simple symbolic calculus, lead to the same conclusion even if the integral equations considered are of the first kind.

A more important result, difficult to establish with classical theories, concerns the regularity of the solution $u(M)$ of the differential system and of the solutions μ^j of the system of integral equations. The simplest one is the following: (a) if ω is the sources' support, u is the C^∞ function in $\Omega - \omega$; (b) if ω is strictly contained in Ω, then the μ^j are C^∞ functions on Γ. These important regularity properties can be helpful for improving numerical methods.

2. INTEGRAL EQUATIONS FOR THE SCALAR HELMHOLTZ EQUATION

Starting from the partial differential formulation of linear acoustics in homogeneous media (see, for example, reference [7] for the establishment of the linearized governing equations), the integral representation of the scattered and diffracted field is introduced and the corresponding integral equations are recalled. Emphasis is given to some delicate cases, such as exterior problems for which eigenfrequencies occur in the integral equations, or thin screen diffraction problems with Neumann boundary conditions.

2.1. HELMHOLTZ EQUATION AND CLASSICAL BOUNDARY CONDITIONS

2.1.1. *Statement of the problem in bounded domains*

Let Ω be a bounded domain of space $\mathbb{R}^n$, and Γ be its boundary, assumed to be regular (C^∞ boundary); let $\mathbf{n}$ be the unitary vector normal to Γ and pointing out to $\complement\bar{\Omega}$, the space complementary to $\bar{\Omega}$. If f is a function defined for all M in Ω, one may denote by $\mathrm{Tr} f$ and $\mathrm{Tr}\,\partial_n f$ the functions defined on Γ as the limits

$$\mathrm{Tr} f(P) = \lim_{M \in \Omega \to P \in \Gamma} f(M), \tag{1}$$

$$\mathrm{Tr}\,\partial_n f(P) = \lim_{M \in \Omega \to P \in \Gamma} \mathbf{n}(P).\mathbf{grad} f(M). \tag{2}$$

One looks for the solution of the following boundary value problem:

$$(\Delta + k^2)\varphi = f, \qquad \forall\, M \in \Omega, \tag{3}$$

$$\alpha\,\mathrm{Tr}\,\varphi + \beta\,\mathrm{Tr}\,\partial_n \varphi = 0, \qquad \forall\, M \in \Gamma, \tag{4}$$

where f is any function (or, more generally, distribution) compactly supported in Ω: that is, a function which is zero outside a bounded domain contained in Ω; α and β are C^∞ functions defined on Γ.

Remarks. (i) If $\alpha = 0$ and $\beta = 1$, one gets the Neumann boundary condition; the Dirichlet one is obtained for $\alpha = 1$ and $\beta = 0$.

(ii) The cases of piecewise C^∞ boundary, or piecewise C^∞ boundary conditions, can be studied as limits of C^∞ boundaries, or C^∞ boundary conditions.

(iii) More general boundary conditions can be significant from a physical point of view: for example, an integro-differential boundary condition takes into account the sound propagation within the boundary, but the classical theories cannot prove that such a problem is well-posed—the pseudo-differential operators and the Poisson pseudo-kernel theories are needed.

It can be proved that the problem specified by equations (3) and (4) has one and only one solution unless k belongs to a denumerable sequence k_n; the k_n are called the eigenvalues of the Laplace operator with respect to the domain Ω and the boundary condition (4). If $k = k_n$, then the homogeneous problem ($f \equiv 0$) has a finite number of linearly independent solutions, φ_n, called eigenfunctions; the non-homogeneous problem has no solution unless f is orthogonal to the corresponding eigensolutions of the transposed boundary value problem. This theorem can be established with the Green formula representation of the diffracted field which leads to an integral equation which is shown to be of Fredholm type.

2.1.2. *Unbounded domains: uniqueness conditions*

If Ω is an unbounded domain, it is well known that equation (3) and boundary condition (4) do not determine a unique solution. A third equation is needed, and it is possible to choose between three families of such equations, as follows.

(a) *The Sommerfeld conditions.* These describe the asymptotic behaviour of the solution at infinity (see reference [8]). If r is the distance from M to the origin, φ must satisfy one of the two following relationships:

$$\left.\begin{aligned} &\lim_{r\to\infty} \varphi = O(r^{-(n-1)/2}) \\ &\lim_{r\to\infty} (\partial_r \varphi - ik\varphi) = o(r^{-(n-1)/2}) \end{aligned}\right\}, \tag{5}$$

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$o(r^{-(n-1)/2})$ means that the
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$$\left.\begin{aligned}&\lim_{r\to\infty}\varphi = O(r^{-(n-1)/2})\\&\lim_{r\to\infty}(\partial_r\varphi + ik\varphi) = o(r^{-(n-1)/2})\end{aligned}\right\}, \tag{6}$$

where $O(r^{-(n-1)/2})$ means that the function decreases at least as fast as $r^{-(n-1)/2}$ and $o(r^{-(n-1)/2})$ means that the function decreases faster than $r^{-(n-1)/2}$. Either of the two conditions (5) or (6) will ensure the uniqueness of the solution.

(b) *The limit amplitude principle.* Define $\psi_+(t, M)$ and $\psi_-(t, M)$ as the solutions, satisfying boundary condition (4), of the following initial boundary value problems:

$$\left.\begin{aligned}&\left(\Delta - \frac{1}{c^2}\partial_{tt}^2\right)\psi_-(t, M) = Y(t) f(M)\,\mathrm{e}^{-\mathrm{i}\omega t}, \qquad k^2 = \omega^2/c^2\\&\psi_- = \partial_t\psi_- = 0 \qquad \text{for } t < 0\\&Y(t) = 0 \qquad \text{for } t < 0, \qquad = 1 \qquad \text{for } t > 0,\end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned}&\left(\Delta - \frac{1}{c^2}\partial_{tt}^2\right)\psi_+(t, M) = Y(t) f(M)\,\mathrm{e}^{+\mathrm{i}\omega t}\\&\psi_+ = \partial_t\psi_+ = 0 \qquad \text{for } t < 0\end{aligned}\right\}. \tag{8}$$

The different limits,

$$\varphi_-(M) = \lim_{t\to\infty}\psi_-(t, M) \qquad \text{and} \tag{9}$$

$$\varphi_+(M) = \lim_{t\to\infty}\psi_+(t, M) \tag{10}$$

are then uniquely determined solutions of the general boundary value problem specified by equations (3) and (4).

(c) *The limit absorption principle.* Define, for $\varepsilon > 0$, the functions $\varphi_{-\varepsilon}$ and $\varphi_{+\varepsilon}$ as the unique bounded solutions of equations (3) and (4), where k^2 is replaced by $(k - \mathrm{i}\varepsilon)^2$ or $(k + \mathrm{i}\varepsilon)^2$. Then the two different limits

$$\varphi_{+0} = \lim_{\varepsilon\to 0}\varphi_{+\varepsilon} \qquad \text{and} \tag{11}$$

$$\varphi_{-0} = \lim_{\varepsilon\to 0}\varphi_{-\varepsilon} \tag{12}$$

exist and are unique, and they are solutions of equations (3) and (4) with real k.

It has been proved [9, 10] that equations (5), (9) and (11) define the same solution, while equations (6), (10) and (12) define another unique solution. All these conditions are various expressions of the energy conservation principle from which the Helmholtz equation is derived. Throughout this paper, the Sommerfeld condition (5) (or another equivalent condition) is assumed when necessary.

2.2. ELEMENTARY SOLUTION FOR THE HELMHOLTZ EQUATION; LAYER POTENTIALS; INTEGRAL EQUATIONS

Starting from the concept of an elementary solution of the Helmholtz equation, it is possible to define a class of functions which are solutions of the homogeneous Helmholtz equation in an open domain: the simple layer and double layer potentials. When the non-homogeneous equation is to be solved, the use of a layer potential enables one to construct an integral equation defined on the boundary of the propagation domain.

2.2.1. *Elementary solutions and elementary kernels for the Helmholtz operator*

A function (or more precisely a distribution) G is an elementary solution of the Helmholtz equation if it satisfies

$$(\Delta + k^2)G = \delta. \tag{13}$$

G represents the sound field at a point M due to a spherical point source located at the origin. If the point source is located at S, because of the fact that the Helmholtz equation is a convolution one, the field at any point M will be obtained by a translation of G; that is to say that the value of G at M depends on the distance $r(M,S)$ only.

It is to be noticed that a function G, satisfying equation (13) is not necessarily unique. But if one adds that G must represent radiation into a free field, then G will satisfy the Sommerfeld condition and will be unique. One then has the well known results (for the $e^{-i\omega t}$ time-dependency convention):

$$G(M,S) = -\frac{e^{ikr(M,S)}}{4\pi r(M,S)} \quad \text{in } \mathbb{R}^3, \tag{14}$$

$$G(M,S) = -\frac{i}{4}H_0[kr(M,S)] \quad \text{in } \mathbb{R}^2. \tag{15}$$

Here H_0 is the Hankel function of the first kind. In reference [6], a proof of results (14) and (15), based on the limiting absorption principle, can be found.

2.2.2. *Simple layer and double layer potentials*

Before defining these potentials, the expression for the Laplace operator in the distribution sense is needed.

Let φ be a function, defined in the whole $\mathbb{R}^3$-space, discontinuous along a closed surface Γ, and having a normal derivative discontinuous along Γ. The surface Γ divides the space into two domains, Ω and $\complement\bar{\Omega}$, and the unitary vector $\mathbf{n}$ normal to Γ is assumed to point out to $\complement\Omega$. One may introduce the following definitions:

$$\left.\begin{aligned} \mathrm{Tr}^+\varphi &= \lim_{M\in\complement\bar{\Omega}\to M\in\Gamma} \varphi(M) \\ \mathrm{Tr}^-\varphi &= \lim_{M\in\Omega\to P\in\Gamma} \varphi(M) \end{aligned}\right\}, \tag{16}$$

$$\left.\begin{aligned} \mathrm{Tr}^+\partial_n\varphi &= \lim_{M\in\complement\bar{\Omega}\to P\in\Gamma} \partial_n\varphi(M) \\ \mathrm{Tr}^-\partial_n\varphi &= \lim_{M\in\Omega\to P\in\Gamma} \partial_n\varphi(M) \\ \partial_n\varphi &= \mathbf{n}.\mathbf{grad}\,\varphi \end{aligned}\right\}. \tag{17}$$

It is impossible to define the Laplacian of φ in the classical way, because this function is not differentiable in the vicinity of Γ. It is first necessary to associate with φ a distribution, again denoted by φ (for definitions see reference [6], or [11] for example). Denoting by $\{\Delta\varphi\}$ the classical Laplacian of φ which is assumed to be defined everywhere but on Γ, and by $\Delta\varphi$ the Laplacian in the distribution sense, one can show that

$$\Delta\varphi = \{\Delta\varphi\} + \frac{d}{dn}[(\mathrm{Tr}^+\varphi - \mathrm{Tr}^-\varphi)\delta_\Gamma] + (\mathrm{Tr}^+\partial_n\varphi - \mathrm{Tr}^-\partial_n\varphi)\delta_\Gamma. \tag{18}$$

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In equation (18), δ_Γ is the Dirac measure with support Γ; $(\mathrm{d}/\mathrm{d}n)[(\mathrm{Tr}^+\varphi - \mathrm{Tr}^-\varphi)\delta_\Gamma]$ can be considered as the normal derivative of the distribution $(\mathrm{Tr}^+\varphi - \mathrm{Tr}^-\varphi)\delta_\Gamma$, the support of which is Γ.

Assume now that φ satisfies the homogeneous Helmholtz equation in Ω as well as in $\complement\bar{\Omega}$. One then obtains

$$(\Delta + k^2)\varphi = \{(\Delta + k^2)\varphi\} + \frac{\mathrm{d}}{\mathrm{d}n}[(\mathrm{Tr}^+\varphi - \mathrm{Tr}^-\varphi)\delta_\Gamma] + (\mathrm{Tr}^+\partial_n\varphi - \mathrm{Tr}^-\partial_n\varphi)\delta_\Gamma$$

$$= \frac{\mathrm{d}}{\mathrm{d}n}[(\mathrm{Tr}^+\varphi - \mathrm{Tr}^-\varphi)\delta_\Gamma] + (\mathrm{Tr}^+\partial_n\varphi - \mathrm{Tr}^-\partial_n\varphi)\delta_\Gamma. \tag{19}$$

Equation (19) shows that the distribution φ satisfies, in the whole space, a non-homogeneous Helmholtz equation. The second term of the right-hand side,

$$(\mathrm{Tr}^+\partial_n\varphi - \mathrm{Tr}^-\partial_n\varphi)\delta_\Gamma = \mu\,\delta_\Gamma,$$

represents a layer of simple sources, the density of which is μ. The first term,

$$\frac{\mathrm{d}}{\mathrm{d}n}[(\mathrm{Tr}^+\varphi - \mathrm{Tr}^-\varphi)\delta_\Gamma] = \frac{\mathrm{d}}{\mathrm{d}n}(\nu\,\delta_\Gamma),$$

is a layer of dipole sources with density ν. Consider now expressions for the radiation of such sources.

(a) *The simple layer potential* φ_1 is the potential due to a layer of simple sources; consequently φ_1 satisfies (in the distribution sense) the non-homogeneous equation

$$(\Delta + k^2)\varphi_1 = \mu\,\delta_\Gamma. \tag{20}$$

Because equation (20) is of the convolution type, one has $\varphi_1 = G*\mu\delta_\Gamma$ (where $*$ indicates the convolution product): that is (see reference [6, or 11]),

$$\varphi_1(M) = \int_\Gamma \mu(P)\,G(M,P)\,\mathrm{d}P, \tag{21}$$

where G is the free field elementary solution. By comparing equations (19) and (20), it can be seen that φ_1 is a continuous function, but its normal derivative has a discontinuity

$$\mathrm{Tr}^+\partial_n\varphi_1 - \mathrm{Tr}^-\partial_n\varphi_1 = \mu. \tag{22}$$

Furthermore the following equalities can be proved:

$$\left.\begin{aligned} \mathrm{Tr}^+\partial_{n(P_0)}\varphi_1(P_0) &= \frac{\mu(P_0)}{2} + \int_\Gamma \mu(P)\,\partial_{n(P_0)}G(P,P_0)\,\mathrm{d}P \\ \mathrm{Tr}^-\partial_{n(P_0)}\varphi_1(P_0) &= -\frac{\mu(P_0)}{2} + \int_\Gamma \mu(P)\,\partial_{n(P_0)}G(P,P_0)\,\mathrm{d}P \end{aligned}\right\}. \tag{23}$$

(b) *The double layer potential* φ_2, that due to a layer of dipole sources, is a solution of the non-homogeneous equation

$$(\Delta + k^2)\varphi_2 = \frac{\mathrm{d}}{\mathrm{d}n}(\mu\,\delta_\Gamma). \tag{24}$$

Here again it is obvious that

$$\varphi_2 = G * \frac{\mathrm{d}}{\mathrm{d}n}(\mu\,\delta_\Gamma),$$

or, more explicitly,

$$\varphi_2(M) = -\int_\Gamma \mu(P)\,\partial_{n(P)} G(M,P)\,\mathrm{d}P. \tag{25}$$

By comparing equations (19) and (24), it is seen that φ_2 has a continuous normal derivative, but is a discontinuous function such that

$$\mathrm{Tr}^+\varphi_2 - \mathrm{Tr}^-\varphi_2 = \mu,$$

the following equalities being easily proved:

$$\left.\begin{aligned} \mathrm{Tr}^+\varphi_2(P_0) &= \frac{\mu(P_0)}{2} - \int_\Gamma \mu(P)\,\partial_{n(P)} G(P_0,P)\,\mathrm{d}P \\ \mathrm{Tr}^-\varphi_2(P_0) &= -\frac{\mu(P_0)}{2} - \int_\Gamma \mu(P)\,\partial_{n(P)} G(P_0,P)\,\mathrm{d}P \end{aligned}\right\}. \tag{26}$$

It is important to remark that the value of $\partial_n\varphi_2$ at any point $P_0 \in \Gamma$ cannot be expressed by a classical integral: in fact, to get such an expression, the function G has to be differentiated twice, so giving a non-integrable function. Nevertheless, a meaningful integral expression can be obtained by introducing the concept of the finite part of an integral in a sense close to that defined by Hadamard [6, or 11]; in the present case, the only possible definition of that finite part is given by the limit

$$\begin{aligned} \partial_{n(P_0)}\varphi_2(P_0) &= -\mathrm{Pf.}\int_\Gamma \mu(P)\,\partial_{n(P_0)}\partial_{n(P)} G(P_0,P)\,\mathrm{d}P \\ &= -\lim_{M\to P_0}\int_\Gamma \mu(P)\,\partial_{n(P_0)}\partial_{n(P)} G(M,P)\,\mathrm{d}P. \end{aligned} \tag{27}$$

In what follows, each time that the symbol Pf. is used it is to be understood as the limit value of the integral. Nevertheless, despite the non-integrability of the kernel, the numerical evaluation of such an integral can be easily done by using rather classical approximations, as will be shown in section 3.

2.2.3. *Integral equations*

Let Γ be a closed surface in the $\mathbb{R}^3$ space (or a closed curve in the $\mathbb{R}^2$ space), the inside of which is Ω and the outside $\complement\bar{\Omega}$. One looks for a function, φ, which is a solution of the Helmholtz equation

$$(\Delta + k^2)\varphi = f, \tag{28}$$

where f is a source with bounded support and k is the wave number, assumed to be real, and which satisfies on Γ one of the two following boundary conditions:

$$\left.\begin{aligned} &\mathrm{Tr}\,\varphi = 0, && \text{Dirichlet problem} \\ &\mathrm{Tr}\,\partial_n\varphi = 0, && \text{Neumann problem} \end{aligned}\right\}. \tag{29}$$

In equations (29), the symbol Tr stands for Tr^+ or Tr^-, depending on whether an exterior problem or an interior one is being considered. One may denote by D_e and D_i, respectively,

the exterior and interior Dirichl
exterior and interior Neumann p
condition must be added:

$\lim_{r\to\infty} \varphi = O(r^{-(n-}$

$\lim_{r\to\infty} (\partial_r\varphi - ik\varphi)$

The conditions of existence and u
are well known; they can be obt
equations derived from the Gree
how this theorem can be used).

Let G be the elementary solu
condition. One seeks a solution

$\varphi =$

(α and β are constants), or, more

$\varphi(M) = \varphi_0(M) +$

$\varphi_0 = G * f$

The functions μ and ν must be s
for the D_i problem,

$\alpha\int_\Gamma \mu(P)\,G(P_0,P)\,\mathrm{d}P - \frac{\beta\nu(P}{2}$

for the D_e problem,

$\alpha\int_\Gamma \mu(P)\,G(P_0,P)\,\mathrm{d}P + \frac{\beta\nu(P}{2}$

for the N_i problem,

$-\frac{\alpha\mu(P_0)}{2} + \alpha\int_\Gamma \mu(P)\,\partial_{n(P_0)} G(P_0,P$

and, for the N_e problem,

$\frac{\alpha\mu(P_0)}{2} + \alpha\int_\Gamma \mu(P)\,\partial_{n(P_0)} G(P_0,P)$

First of all, a second relation b
solution of the equations (33) to
For the choice $\mu = \nu$ (which is th
proved.

Theorem. For the interior
solution, whatever α and β

the exterior and interior Dirichlet problems; N_e and N_i similarly denote, respectively, the exterior and interior Neumann problems. If an exterior problem is dealt with, a Sommerfeld condition must be added:

$$\left.\begin{array}{l}\lim\limits_{r\to\infty} \varphi = O(r^{-(n-1)/2}) \\ \lim\limits_{r\to\infty} (\partial_r \varphi - \mathrm{i}k\varphi) = o(r^{-(n-1)/2}), \qquad n = \text{space dimension}\end{array}\right\}. \tag{30}$$

The conditions of existence and uniqueness of the solutions of equations (28), (29) and (30) are well known; they can be obtained by applying the Fredholm alternative to the integral equations derived from the Green formula (see reference [5], for example, where it is shown how this theorem can be used).

Let G be the elementary solution of the Helmholtz equation satisfying the Sommerfeld condition. One seeks a solution φ of the form

$$\varphi = G*f + G*\left[\alpha\mu\,\delta_\Gamma + \beta\frac{\mathrm{d}}{\mathrm{d}n}(\nu\,\delta_\Gamma)\right] \tag{31}$$

(α and β are constants), or, more explicitly,

$$\left.\begin{array}{l}\varphi(M) = \varphi_0(M) + \int_\Gamma [\alpha\mu(P)\,G(M,P) - \beta\nu(P)\,\partial_{n(P)}G(M,P)]\,\mathrm{d}P \\ \varphi_0 = G*f\end{array}\right\}. \tag{32}$$

The functions μ and ν must be such that the boundary condition is satisfied, which leads to, for the D_i problem,

$$\alpha\int_\Gamma \mu(P)\,G(P_0,P)\,\mathrm{d}P - \frac{\beta\nu(P_0)}{2} - \beta\int_\Gamma \nu(P)\,\partial_{n(P)}G(P_0,P)\,\mathrm{d}P = -\varphi_0(P_0), \qquad \forall\, P_0 \in \Gamma, \tag{33}$$

for the D_e problem,

$$\alpha\int_\Gamma \mu(P)\,G(P_0,P)\,\mathrm{d}P + \frac{\beta\nu(P_0)}{2} - \beta\int_\Gamma \nu(P)\,\partial_{n(P)}G(P_0,P)\,\mathrm{d}P = -\varphi_0(P_0), \qquad \forall\, P_0 \in \Gamma, \tag{34}$$

for the N_i problem,

$$-\frac{\alpha\mu(P_0)}{2} + \alpha\int_\Gamma \mu(P)\,\partial_{n(P_0)}G(P_0,P)\,\mathrm{d}P - \beta\,\mathrm{Pf.}\int_\Gamma \nu(P)\,\partial_{n(P_0)}\partial_{n(P)}G(P_0,P)\,\mathrm{d}P = -\partial_{n(P_0)}\varphi(P_0), \qquad \forall\, P_0 \in \Gamma, \tag{35}$$

and, for the N_e problem,

$$\frac{\alpha\mu(P_0)}{2} + \alpha\int_\Gamma \mu(P)\,\partial_{n(P_0)}G(P_0,P)\,\mathrm{d}P - \beta\,\mathrm{Pf.}\int_\Gamma \nu(P)\,\partial_{n(P_0)}\partial_{n(P)}G(P_0,P)\,\mathrm{d}P = -\partial_{n(P_0)}\varphi_0(P_0), \qquad \forall\, P_0 \in \Gamma. \tag{36}$$

First of all, a second relation between μ and ν is necessary to ensure the uniqueness of the solution of the equations (33) to (36) (note, however, that the existence is not thereby proved). For the choice $\mu = \nu$ (which is the simplest relation), the following important theorem can be proved.

Theorem. For the interior problems D_i and N_i, equations (33) and (35) have a unique solution, whatever α and β are, unless k is an eigenvalue of the problem; in such a case,

- the homogeneous equation (33) or (35) has a finite number of independent solutions. For the exterior problems D_e and N_e, the equations (34) and (36) have a unique solution for any k if the ratio α/β has a non-zero imaginary part.

This result shows that, in any case, the solution of the interior or exterior Dirichlet and Neumann problems can be represented in the form

$$\varphi(M) = \varphi_0(M) + \int_\Gamma \mu(P)\,[\alpha G(M,P) - \beta\,\partial_{n(P)} G(M,P)]\,\mathrm{d}P. \tag{37}$$

The particular case of the representation derived from Green's formula is obtained with

$$\alpha = 1, \quad \beta = 0, \quad \mu = \operatorname{Tr} \partial_n \varphi \quad \text{for Dirichlet problems,}$$
$$\alpha = 0, \quad \beta = 1, \quad \mu = \operatorname{Tr} \varphi \quad \text{for Neumann problems.}$$

Remark. For exterior problems, the integral representation derived from Green's formula is not convenient for numerical computation. Indeed, the integral equation so obtained has the eigenvalues of the D_i problem if the D_e one is solved, or of the N_i problem if the N_e one is solved. Nevertheless, it can be shown that these equations always have a unique solution, but it is always difficult to compute accurately the solution of an equation which has a non-trivial solution for a zero second member.

Proof of the theorem. The proof of the theorem is presented here for one case only: that of the representation of the solution of the D_i problem with the help of a double layer potential. The integral equation obtained here is

$$-\frac{\mu(P_0)}{2} - \int_\Gamma \mu(P)\,\partial_{n(P)} G(P_0,P)\,\mathrm{d}P = -\varphi_0(P_0), \qquad \forall\, P_0 \in \Gamma. \tag{38}$$

Assume first that k is not an eigenvalue of problem D_i; then let ψ be its unique solution, and define the function ϕ as the solution of the following N_e problem:

$$\left.\begin{array}{ll} (\Delta + k^2)\,\phi = 0 & \text{in } \complement\bar{\Omega} \\ \operatorname{Tr}^+ \partial_n \phi = \operatorname{Tr}^-(\partial_n \psi - \partial_n \varphi^0) & \text{on } \Gamma \\ + \text{ Sommerfeld condition} & \end{array}\right\}.$$

The function μ defined on Γ by

$$\mu = \operatorname{Tr}^-(\psi - \varphi_0) - \operatorname{Tr}^+ \phi = -\operatorname{Tr}^- \varphi_0 - \operatorname{Tr}^+ \phi \tag{39}$$

enables one to construct the double layer potential

$$\varphi_d(M) = -\int_\Gamma \mu(P)\,\partial_{n(P)} G(M,P)\,\mathrm{d}P.$$

One can now show that φ_d satisfies the boundary condition $\operatorname{Tr}^- \varphi_d = -\varphi_0$ on Γ: that is to say that μ given by equation (39) is a solution of equation (38). For that purpose, one may construct the function

$$U = Y(\Omega)\,[\psi - \varphi_0] + [1 - Y(\Omega)]\,\phi - \varphi_d, \tag{40}$$

where

$$Y(\Omega) \begin{cases} = 1 & \text{for } M \in \Omega, \\ = 0 & \text{for } M \in \complement\bar{\Omega}. \end{cases}$$

The function U is continuous across Γ because the jump in φ_d is equal to that of $Y(\Omega)[\psi - \varphi_0] + [1 - Y(\Omega)]\phi$; its normal derivative is equally continuous. As a consequence, U is a solution of the homogeneous Helmholtz equation defined in the whole space, because

$$(\Delta + k^2)\, Y(\Omega)[\psi - \varphi_0] = +\frac{\mathrm{d}}{\mathrm{d}n}(\mathrm{Tr}^- \varphi_0\, \delta_\Gamma) - \mathrm{Tr}^- \partial_n(\psi - \varphi_0)\,\delta_\Gamma,$$

$$(\Delta + k^2)[1 - Y(\Omega)]\phi = \frac{\mathrm{d}}{\mathrm{d}n}(\mathrm{Tr}^+ \phi\, \delta_\Gamma) + \mathrm{Tr}^+ \partial_n \phi\, \delta_\Gamma,$$

$$(\Delta + k^2)\varphi_d = \frac{\mathrm{d}}{\mathrm{d}n}(\mu\, \delta_\Gamma) = -\frac{\mathrm{d}}{\mathrm{d}n}[(\mathrm{Tr}^- \varphi_0 + \mathrm{Tr}^+ \phi)\,\delta_\Gamma].$$

Furthermore, U satisfies the Sommerfeld condition; this finally implies $U \equiv 0$ in the whole space, and, consequently,

$$\varphi_d = \psi - \varphi_0 \quad \text{in } \Omega, \qquad \text{or } \mathrm{Tr}^- \varphi_d = -\mathrm{Tr}^- \varphi_0 \quad \text{on } \Gamma.$$

One can now prove that μ as defined in equation (39) is the only solution of equation (38). Assume that a second solution μ' exists. Then, the function $\varphi_d - G*(\mathrm{d}/\mathrm{d}n)(\mu'\delta_\Gamma)$, which is zero in Ω, has a continuous gradient, satisfies the homogeneous Helmholtz equation in Ω and $\complement\bar{\Omega}$, and the Sommerfeld condition at infinity, and has a zero normal gradient on Γ; hence it is identically zero in $\complement\bar{\Omega}$ as in Ω, and, consequently, its jump $\mu - \mu'$ is zero. Finally, if k is an eigenvalue of the D_i problem, there exists a finite number of linearly independent eigenfunctions ψ_j. The functions

$$\mu_j = -\mathrm{Tr}^+ \phi_j, \qquad \phi_j \text{ defined by } \mathrm{Tr}^+ \partial_n \phi_j = \mathrm{Tr}^- \psi_j \tag{41}$$

are solutions of the homogeneous integral equation associated with equation (38). It can be shown that the functions μ_j are linearly independent; furthermore, the only linearly independent solutions of the homogeneous equation associated with equation (38) are those defined by equation (41). Thus the theorem is proved.

2.3. INTEGRAL REPRESENTATION OF THE FIELD DIFFRACTED BY AN INFINITELY THIN SCREEN

From a physical point of view, the concept of an infinitely thin screen is meaningless: indeed, a screen of zero thickness does not exist. But it is interesting to look for possible mathematical simplifications when such a screen is considered; furthermore, because of the continuity of the diffracted field with respect to the diffracting obstacle geometry, if a zero-thickness diffracting obstacle is mathematically meaningful, the diffracted field so obtained will be close to that due to a thin screen, as shown in references [12] and [13]. Accordingly, for the physical problem, the mathematical infinitely thin screen has to be defined as a limiting case of a finitely thin obstacle. We will give an idea of the limiting procedure which can be used for that purpose.

Let us consider, in $\mathbb{R}^2$, a sequence of obstacles Ω^ε limited by a boundary Γ^ε, and which have a curve segment Γ as limit when $\varepsilon \to 0$ (for example, the successive ellipses of the elliptic co-ordinate system have such a limit). Assume, for example, that Ω^ε diffracts an incident wave and that the total field satisfies a constant boundary condition whatever ε is. For sufficiently large r, the diffracted field φ_d^ε has the asymptotic development

$$\varphi_d^\varepsilon \simeq \frac{\mathrm{e}^{ikr}}{\sqrt{r}} \sum_{-\infty}^{+\infty} a_n^\varepsilon \mathrm{e}^{in\theta}.$$

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The energy radiated through the circle at infinity is proportional to

$$E^{\varepsilon}=\sum_{-\infty}^{+\infty}|a_n^{\varepsilon}|^2$$

and is bounded by the incident energy flux. So, as $\varepsilon \to 0$, the sequence E^{ε} remains bounded, and a classical theorem ensures that the a_n^{ε} have limits a_n. To these limits a_n corresponds a φ_d of the form

$$\varphi_d \simeq \frac{e^{ikr}}{\sqrt{r}}\sum_{-\infty}^{+\infty} a_n e^{in\theta} \qquad \text{for large } r.$$

This function being analytic in a non-zero domain, it has a unique continuation up to Γ on which the initial boundary condition is satisfied by the total field. It is important to remark that no edge conditions appear.

If the Green integral representation of the φ_d^{ε} is considered, it can be shown that this integral representation has a limit. Consequently, another correct statement of the problem is the following.

Let Γ be a bounded two- (or one-) dimensional domain in $\mathbb{R}^3$ (or $\mathbb{R}^2$) with normal n. One seeks a solution of the Helmholtz equation

$$(\Delta + k^2)\varphi = f \qquad \text{in } \complement\Gamma.$$

The function φ satisfies a Sommerfeld condition, and has an integral representation of the form

$$\varphi(M)=\varphi_0(M)+\int_{\Gamma}\{\mu(P)\,G(M,P)-\nu(P)\,\partial_{n(P)}G(M,P)\}\,dP.$$

(a) *Dirichlet boundary condition.* The total field must be a continuous function in the whole space, which implies that $\nu = 0$. The function μ is the unique solution of

$$\int_{\Gamma}\mu(P)\,G(P_0,P)\,dP=-\varphi_0(P_0), \qquad \forall\, P_0 \in \Gamma. \tag{42}$$

The discontinuity of the normal gradient of φ is μ.

(b) *Neumann boundary condition.* The total field must have a gradient continuous in the whole space, which implies that $\mu = 0$. The function ν is the unique solution of

$$\text{Pf.}\int_{\Gamma}\nu(P)\,\partial_{n(P_0)}\partial_{n(P)}G(P_0,P)\,dP=-\partial_{n(P_0)}\varphi_0(P_0), \qquad \forall\, P_0\in\Gamma. \tag{43}$$

Remark. The so-called "edge conditions" are useless when the screen is defined as a limiting case, or when an integral representation of the diffracted field is adopted. Furthermore, when the classical partial differential equation and boundary condition are used, the edge conditions appear as sufficient conditions only to secure the uniqueness of the solution [14]. In reference [14] it is proved that the integral equation (42) provides a solution satisfying these conditions. It is consequently better to speak of "edge properties".

3. NUMERICAL SOLUTION OF THE INTEGRAL EQUATIONS ASSOCIATED WITH THE SCALAR HELMHOLTZ EQUATION

The present section deals with one numerical method only which appears to be simple but efficient in every case. Three classes of problems have been pointed out in the preceding

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The method is essentially based
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equation (44) becomes

It is obvious that equation (47) can
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can be solved; it is to be noticed th
by an N-step function. The test p
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accuracy of the result.

section: interior problems for which forced oscillations as well as free modes (eigenwavenumbers or eigenfrequencies) have to be computed; exterior problems in which the associated integral equations can have eigenfrequencies; and diffraction by an infinitely thin screen for which the finite part of an integral can occur. Examples will be given for each of these three cases.

The method is essentially based on the approximation of the integral by a sum, and the unknown function by its approximate value at a finite number of points. A linear algebraic system is thus obtained by letting the approximate first member of the equation be equal to the second member at a finite number of points; when eigenwavenumbers are looked for, an approximation is obtained by adjusting the wavenumber so that the determinant of the system is as close to zero as possible.

The numerical approximations here proposed are compared to either analytical solutions (variable separation), or experiments. The good agreement shows the efficiency of the method.

3.1. GENERAL DESCRIPTION OF THE APPROXIMATION SCHEME

In sections 2.2 and 2.3 several integral equations have been established. They can be written in the form

$$K\mu = f, \tag{44}$$

where K is an integral operator of Fredholm type of the first kind,

$$\mu \to K\mu = \int_\Gamma \bar{K}(P_0,P)\,\mu(P)\,\mathrm{d}P, \qquad P_0 \in \Gamma, \tag{45}$$

or of the second kind,

$$\mu \to K\mu = \mu(P_0) + \int_\Gamma \bar{K}(P_0,P)\,\mu(P)\,\mathrm{d}P, \qquad P_0 \in \Gamma. \tag{46}$$

In these expressions, the integrals are of several kinds: Riemann integrals, Cauchy principal values, Hadamard finite parts. The function $\bar{K}(P_0,P)$, called the kernel of the integral equation, is always a singular function.

Whatever the operator K is, the same approximation scheme is adopted. The boundary Γ is divided into N elements, denoted by Γ_j ($j = 1, 2, \ldots, N$), the centers of which are named P_j; the areas (or lengths in $\mathbb{R}^2$) of the Γ_j are of the same order of magnitude. If the linear dimensions of the Γ_j are small enough compared to the wavelength (less than $\lambda/6$ seems to be sufficient), it is quite reasonable to approximate the unknown function $\mu(P)$ by a constant μ_j in each Γ_j. With μ_j again denoting the function which is zero everywhere on Γ but Γ_j, equation (44) becomes

$$\sum_{j=1}^{N} K\mu_j = f. \tag{47}$$

It is obvious that equation (47) cannot be satisfied everywhere on Γ, because its first member depends on N constants (the μ_j) only. If the equality (47) is satisfied at N points, equation (47) can be solved; it is to be noticed that this is equivalent to the approximation of the function f by an N-step function. The test points must be the P_j if K has the form (46); they can be other points when K is given by expression (45). It is possible to choose more than N test points and to solve the resulting algebraic system in the mean square sense, but such a complexity does not seem to be useful: the computing time is much increased, but not the accuracy of the result.

Finally, the most interesting approximation is provided by a system of the following form:

$$\sum_{j=1}^{N} K\mu_j(P_i) = f(P_i), \qquad i = 1, 2, \ldots, N. \tag{48}$$

Nevertheless, it is generally impossible to obtain an exact expression of the

$$K\mu_j(P_i). \tag{49}$$

In such an expression, the integral of the kernel, extended to Γ_j, occurs. An approximation of the value of such integrals will be needed.

Two cases are to be distinguished.

(a) If $i \neq j$ the kernel $\bar{K}(P_i,P)$ is regular everywhere in Γ_j; the simplest approximation of expression (49) is then provided by

$$\int_{\Gamma_j} \bar{K}(P_i,P)\,\mu_j\,\mathrm{d}P \simeq \mu_j \bar{K}(P_i,P_j) \times \text{area of } \Gamma_j. \tag{50}$$

(b) If $i = j$, the kernel $\bar{K}(P_i,P)$ is singular in P_i, and another kind of approximation is needed. For example, Γ_i can be replaced by a disk, located in the plane tangent to Γ at the point P_i; the area of the disk is chosen equal to that of Γ_i; the analytical integration of $\bar{K}(P_i,P)$ over that disk then provides a good approximation.

Remark. It is sometimes necessary to obtain more accurate estimations of the integral of the kernel than that given by expression (50). A powerful method is the following: let Γ_j be divided into n sub-elements Γ_j^l (4 to 9 are in general enough), centered at points P_l'; an approximation is then given by

$$\int_{\Gamma_j} \bar{K}(P_i,P)\,\mu_j\,\mathrm{d}P \simeq \mu_j \sum_{l=1}^{n} \bar{K}(P_i,P_l') \times \text{area of } \Gamma_j^l. \tag{51}$$

3.2. INTERIOR PROBLEMS

For the two interior problems here proposed, both forced and free oscillations are examined. The first example is that of a plane circular domain on the boundary of which a Neumann condition is assumed. Comparison of the numerical solution to the analytical one is made. The second example deals with an ellipsoidal room containing a sphere, the center of which is that of the ellipsoid; here again, the Neumann boundary condition is considered. The numerical results are compared to experimental ones.

3.2.1. *Circular plane domain: comparison of the numerical method to the analytical one* [15]

Consider a plane circular domain with center O and radius a (see Figure 1); a cylindrical point source is located at S, the cylindrical co-ordinates of which are (R, O). A point M in Ω has cylindrical co-ordinates denoted by r and Θ; the co-ordinates of a boundary point P are a and θ. The acoustical field $\varphi(M)$ due to the point source S satisfies the equation

$$(\Delta + k^2)\,\varphi(M) = \delta_S(M), \qquad \forall\, M \in \Omega, \tag{52}$$

and the boundary condition

$$\mathrm{Tr}\,\partial_n \varphi(P) = 0, \qquad \forall\, P \in \Gamma \tag{53}$$

LAY

Figure 1.

(n is the outer normal). Let the d

$\varphi(M) = \varphi_0($

$\varphi_0(M) = G*$

The boundary condition (53) pr

$-\frac{\mu(P)}{2} - \frac{i}{4}\int_0^{2\pi} \partial_{r_0} H_0[$

$P = (a, \theta), \qquad P_0 = ($

(a) *Analytical solution*

The function $\mu(P)$ can be repr

By using the series expansion of

$H_0[kd(M,$

the Fourier coefficients of μ can

$\mu(P)$

The corresponding expression f

$\varphi(M) = -\frac{i}{4} H_0[$

This series is defined for all valu
that $J_n'(ka) = 0$ are the eigenwa

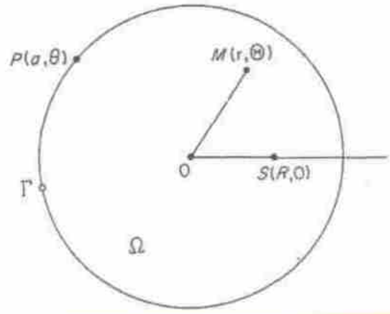


Figure 1. Geometry of the plane circular domain.

(n is the outer normal). Let the diffracted field be represented by a simple layer potential

$$\varphi(M)=\varphi_0(M)+\int_{\Gamma} G(M,P)\,\mu(P)\,\mathrm{d}P,$$

$$\varphi_0(M)=G*f(M),\qquad G(M,P)=-\frac{\mathrm{i}}{4}\mathrm{H}_0[kd(M,P)].$$

The boundary condition (53) provides the integral equation

$$-\frac{\mu(P)}{2}-\frac{\mathrm{i}}{4}\int_0^{2\pi}\partial_{r_0}\mathrm{H}_0[kd(P_0,P)]\,\mu(P)\,a\,\mathrm{d}\theta(P)=-\frac{\mathrm{i}}{4}\partial_{r_0}\mathrm{H}_0[kd(S,P_0)],$$

$$P=(a,\theta),\qquad P_0=(r_0\to a,\theta_0). \tag{54}$$

(a) *Analytical solution*

The function $\mu(P)$ can be represented by a Fourier series

$$\mu(P)=\sum_{m=-\infty}^{+\infty}a_m\,\mathrm{e}^{\mathrm{i}m\theta}.$$

By using the series expansion of $\mathrm{H}_0[kd(M,P)]$,

$$\mathrm{H}_0[kd(M,P)]=\sum_{n=-\infty}^{+\infty}\mathrm{J}_n(kr)\,\mathrm{H}_n(ka)\,\mathrm{e}^{\mathrm{i}n(\Theta-\theta)},\qquad a>r,$$

the Fourier coefficients of μ can easily be calculated, leading to

$$\mu(P)=-\sum_{m=-\infty}^{+\infty}\frac{\mathrm{J}_m(kR)\,\mathrm{H}_m'(ka)}{2\pi a\mathrm{J}_m'(ka)\,\mathrm{H}_m(ka)}\mathrm{e}^{\mathrm{i}m\theta}.$$

The corresponding expression for the sound field is provided by the following series:

$$\varphi(M)=-\frac{\mathrm{i}}{4}\mathrm{H}_0[kd(S,M)]+\frac{\mathrm{i}}{4}\sum_{n=-\infty}^{+\infty}\frac{\mathrm{H}_n'(ka)}{\mathrm{J}_n'(ka)}\mathrm{J}_n(kR)\,\mathrm{J}_n(kr)\,\mathrm{e}^{\mathrm{i}n\Theta}. \tag{55}$$

This series is defined for all values of k but those for which $\mathrm{J}_n'(ka)$ is zero; the values of k such that $\mathrm{J}_n'(ka)=0$ are the eigenwavenumbers of the problem.

(b) *Numerical solution*

Let Γ be divided into N equal arcs Γ_j, with centers P_j. Along each Γ_j, $\mu(P)$ is approximated by a constant. Equation (54) will be approximated by the following linear algebraic system:

$$\sum_{j=1}^{N} A_{ij}\mu_j = f_i, \qquad i = 1, 2, \ldots, N, \tag{56a}$$

$$A_{ij\neq i} = -ik\,H_1(kd_{ij})\cos(\vec{d_{ij}}, \vec{n_i})\,\Gamma_j, \qquad \vec{d_{ij}} = \overrightarrow{P_iP_j}, \tag{56b}$$

$$A_{ii} = 2 - (i\pi\Gamma_i/4a)\,[S_0(z)\,H_1(z) - H_0(z)\,S_1(z)]|_{z=k\Gamma_i/2}, \tag{56c}$$

$$f_i = ik\,H_1(k\rho_i)\cos(\vec{\rho_i}, \vec{n_i}), \qquad \vec{\rho_i} = \overrightarrow{SP_i}. \tag{56d}$$

In equation (56c), A_{ii} is obtained by integration of the kernel of the integral equation over the arc element Γ_ζ, making use of the Struve functions $S_0(z)$ and $S_1(z)$ [16]; $(\vec{d_{ij}}, \vec{n_i})$ is the angle between the vector $\overrightarrow{P_jP_i}$ and the unit vector $\vec{n_i}$ normal to Γ in P_i; $(\vec{\rho_i}, \vec{n_i})$ is the angle between the vector $\overrightarrow{SP_i}$ and $\vec{n_i}$. The total field due to the point source S is approximated by

$$\varphi(M) \simeq \varphi_0(M) - \frac{i}{4}\sum_{j=1}^{N} \mu_j\,H_0[kd(M, P_j)]\,\Gamma_j. \tag{57}$$

Remark. The problem has symmetry with respect to the OS axis, but, to obtain a more significant test of the method's efficiency, the authors of reference [15] ignored this simplification.

(c) *Comparison between numerical and analytical solutions*

In a first computation, eigenwavenumbers were sought; in Table 1 the exact eigenwavenumbers from $ka = 0$ to $ka = 10$ are compared to the approximated ones, for two approximation orders: $N = 20$ and $N = 40$.

In a second calculation, the total approximated field has been computed for $ka = 4{\cdot}5$ and $ka = 18$, along several diameters; the source is located at the point $(R = a/2, 0)$. Figure 2

TABLE 1
Exact (v_t) and approximate (v_a) eigenwavenumbers

	N = 20		N = 40	
v_t	v_a	$\Delta v / v$	v_a	$\Delta v / v$
1·84118	1·84105	7·06 (−5)	1·841165	0·815(−5)
3·05424	3·05393	1·015(−4)	3·05419	1·64 (−5)
3·83171	3·83138	8·61 (−5)	3·83175	1·04 (−5)
4·20119	4·20053	1·57 (−4)	4·20110	2·14 (−5)
5·31755	5·31650	1·97 (−4)	5·31738	3·13 (−5)
5·33144	5·33083	1·14 (−4)	5·33139	0·938(−5)
6·41562	6·41404	2·46 (−4)	6·41579	2·74 (−5)
6·70613	6·70516	1·44 (−4)	6·70600	1·94 (−5)
7·01559	7·01462	1·38 (−4)	7·01542	2·42 (−5)
7·50127	7·49900	3·03 (−4)	7·50097	4·00 (−5)
8·01524	8·01379	1·81 (−4)	8·01503	2·62 (−5)
8·53632	8·53486	1·71 (−4)	8·53609	2·69 (−5)
8·57784	8·57501	3·30 (−4)	8·57748	4·20 (−5)
9·28240	9·28053	2·01 (−4)	9·28209	3·35 (−5)
9·64742	9·64389	3·66 (−4)	9·64695	4·87 (−5)
9·96947	9·96748	2·00 (−4)	9·96917	3·01 (−5)

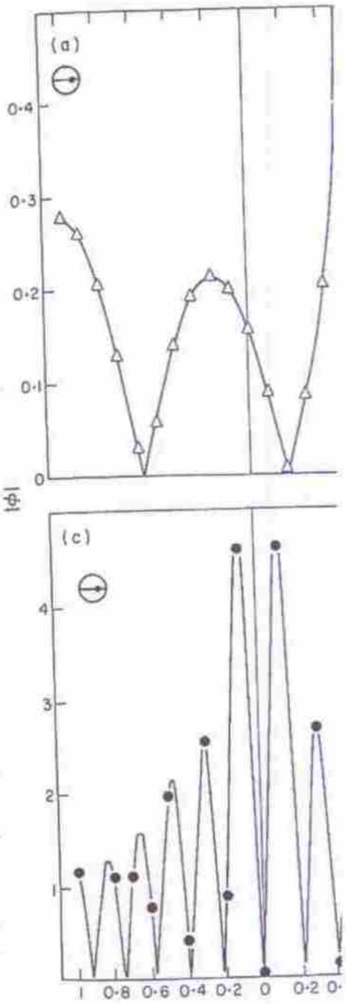


Figure 2. Exact and approximate res
numerical approximations: △, $N = 20$;
through the source and the diameter
(b) $ka = 4{\cdot}5$, $\theta = \pm\pi/6$, $\pi \pm \pi/6$; (c) $ka =$

shows the exact curves on which t
plotted; the approximation order
(d) *Conclusion and remarks*
Table 1 shows that the result
purposes; indeed the accuracy o
furthermore, a 10^{-3} accuracy is q
Figures 2(a) and 2(b) ($ka = 4{\cdot}$
approximation; but when $ka = 18$
accuracy (Figures 2(c) and 2(d)).
that required by physicists: inde
logarithmic representation will s

g each Γ_j, $\mu(P)$ is approximated
llowing linear algebraic system:

N, (56a)

$d_{ij} = \overline{P_i P_j}$, (56b)

$S_1(z)]|_{z=k\Gamma_i/2}$, (56c)

SP_i. (56d)

kernel of the integral equation
ns $S_0(z)$ and $S_1(z)$ [16]; $(d_{ij}, \vec{n}_i)$
$\vec{n}_i$ normal to Γ in P_i; $(\vec{\rho}_i, \vec{n}_i)$ is
d due to the point source S is

$, P_j)]\Gamma_j$. (57)

OS axis, but, to obtain a more
s of reference [15] ignored this

in Table 1 the exact eigenwave-
mated ones, for two approxima-

been computed for $ka = 4{\cdot}5$ and
the point ($R = a/2$, 0). Figure 2

avenumbers

$N=40$	
	$\frac{\Delta v}{v}$
165	0·815(−5)
19	1·64 (−5)
75	1·04 (−5)
10	2·14 (−5)
38	3·13 (−5)
39	0·938(−5)
79	2·74 (−5)
00	1·94 (−5)
42	2·42 (−5)
97	4·00 (−5)
03	2·62 (−5)
09	2·69 (−5)
48	4·20 (−5)
09	3·35 (−5)
95	4·87 (−5)
17	3·01 (−5)

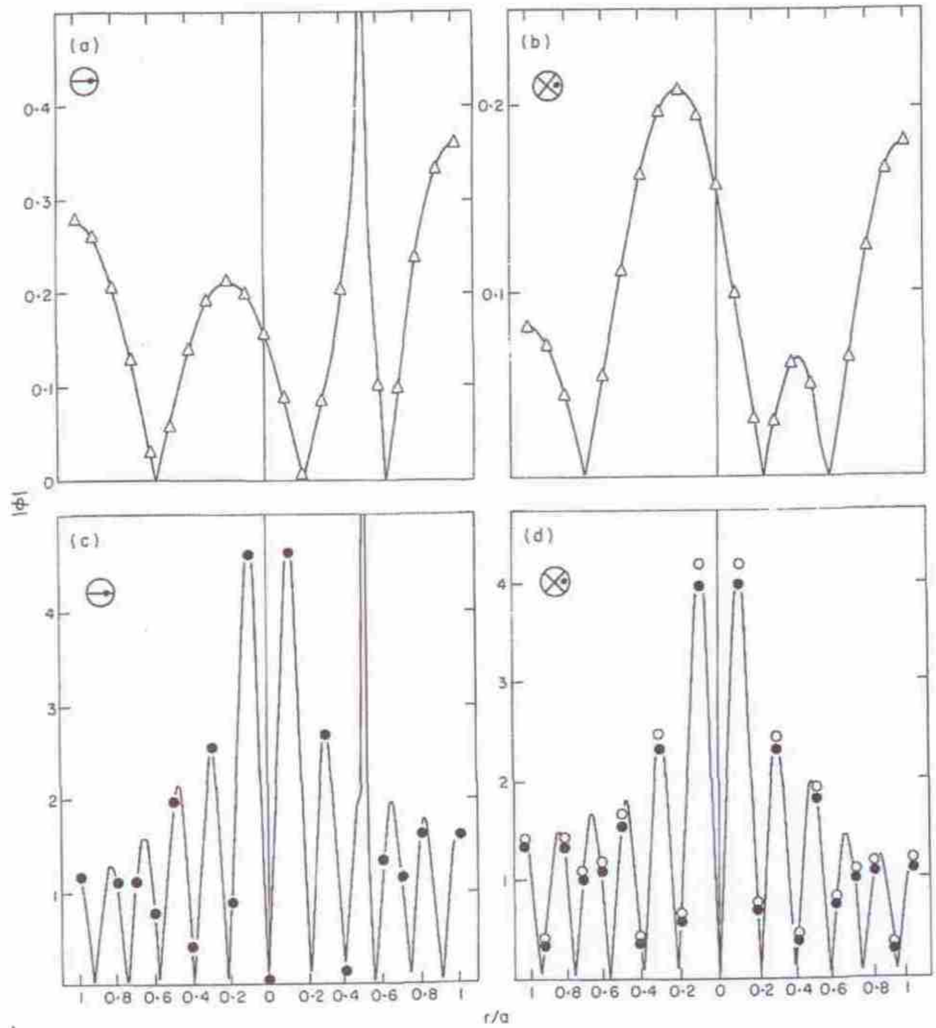


Figure 2. Exact and approximate results for the field in the circular plane domain. ——, Exact solution; numerical approximations: △, $N = 20$; ○, $N = 40$; ●, $N = 80$. θ is the angle between the diameter passing through the source and the diameter along which the acoustic field is computed. (a) $ka = 4{\cdot}5$, $\theta = 0, \pi$; (b) $ka = 4{\cdot}5$, $\theta = \pm\pi/6$, $\pi \pm \pi/6$; (c) $ka = 18$, $\theta = 0, \pi$; (d) $ka = 18$, $\theta = \pm\pi/6$, $\pi \pm \pi/6$.

shows the exact curves on which the numerical values, distinct from the exact ones, have been plotted; the approximation orders used are $N = 20$, $N = 40$ and $N = 80$.

(d) *Conclusion and remarks*

Table 1 shows that the results obtained with $N = 40$ are accurate enough for physical purposes; indeed the accuracy of eigenfrequency measurements is hardly better than 10^{-4}; furthermore, a 10^{-3} accuracy is quite sufficient in most cases.

Figures 2(a) and 2(b) ($ka = 4{\cdot}5$) show that the total field is well described by the $N = 40$ approximation; but when $ka = 18$, an $N = 80$ approximation is needed to obtain an equivalent accuracy (Figures 2(c) and 2(d)). Note that the accuracy of the numerical results is higher than that required by physicists: indeed, one is essentially interested in the dB levels, and such a logarithmic representation will smooth out the errors indicated in Figures 2(a–d).

We conclude this section by a remark concerning a second analytical representation of the solution: that obtained by an eigenfunction series. The authors of reference [15] computed it. It appeared that the computation time is very long (six times that needed by the numerical method), and the accuracy is not as good as that given by the direct numerical approach proposed here.

3.2.2. *Multiply connected domain: comparison between the numerical method and experiment* [17]

Consider an oblate spheroidal domain, containing a sphere in its central region (see Figure 3); the axial lengths of the spheroid are respectively a and $a/2$, the sphere radius being $0{\cdot}4a$. A spherical point sound source is located at one focus of the spheroid. The boundaries Σ and σ of both the ellipsoid and the sphere are assumed to be perfectly reflecting. For such a problem no analytical representation is known. The efficiency of the numerical method will be demonstrated by comparing the prediction so obtained to experimental results.

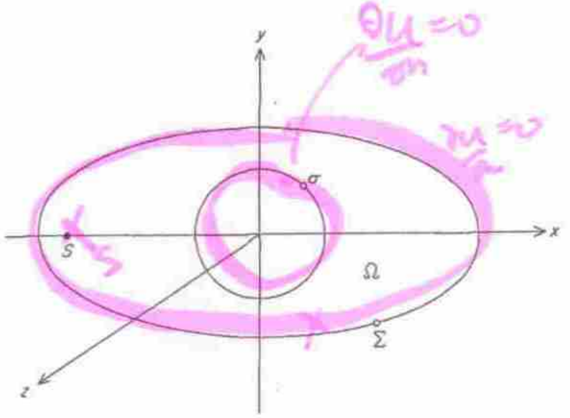


Figure 3. Geometry of the multiply connected domain.

The diffracted field is described by a simple layer potential, the density of which is μ on Σ, and ν on σ. These functions are a solution of the following system of integral equations:

$$-\frac{\mu(P)}{2}+\int_{\Sigma}\mu(P')\,\partial_{n(P)}G(P,P')\,\mathrm{d}P'+\int_{\sigma}\nu(Q')\,\partial_{n(P)}G(P,Q')\,\mathrm{d}Q'=-\partial_{n(P)}\varphi_0(P),\qquad \forall P\in\Sigma,$$

$$\int_{\Sigma}\mu(P')\,\partial_{n(Q)}G(Q,P')\,\mathrm{d}P'-\frac{\nu(Q)}{2}+\int_{\sigma}\nu(Q')\,\partial_{n(Q)}G(Q,Q')\,\mathrm{d}Q'=-\partial_{n(Q)}\varphi_0(Q),\qquad \forall Q\in\sigma, \tag{58}$$

where

$$G(M,P)=-\frac{\mathrm{e}^{\mathrm{i}kr(M,P)}}{4\pi r(M,P)},\qquad \varphi_0(M)=-\frac{\mathrm{e}^{\mathrm{i}kr(S,M)}}{4\pi r(S,M)}.$$

The symmetry of the geometry implies that the functions μ and ν depend on the x co-ordinate only; this simplification has been taken into account. The surfaces Σ and σ have been divided into annular elements (80 for Σ and 32 for σ); on each surface element, μ and ν are approximated by constants μ_i ($i=1, 2, \ldots, 80$) and ν_j ($j=1, 2, \ldots, 32$). The integrals occurring in equations (58) are evaluated by a simple integration formula. Each annular element being divided into 80 sub-elements, the following approximations are used:

$$\int_{\Sigma} \mu(P')\,\partial_{n(P_i)} G(P_i, P')\,\mathrm{d}P' \simeq \sum_{j=1}^{80} \mu_j \left\{ \sum_{k=1}^{80} \partial_{n(P_i)} G(P_i, P'_{jk})\,\Sigma_{jk} + A_{ii}\,\delta_{ij} \right\},$$

$$A_{ii} = \sum_{k=2}^{80} \partial_{n(P_i)} G(P_i, P'_{ik})\,\Sigma_{ik} + \partial_{n(P_i)} G(P_i, \overline{P_{i1}})\,\Sigma_{i1}, \qquad \delta_{ij} = 1 \quad \text{if } i = j, \qquad = 0 \quad \text{if } i \neq j,$$

Σ_{jk} $(j, k = 1, 2, \ldots, 80)$ = areas of the 80 sub-elements of the 80 annular elements of Σ,
P_i $(i = 1, 2, \ldots, 80)$ = center of the sub-element $(i, 1)$,
P'_{jk} $(j, k = 1, 2, \ldots, 80)$ = center of the sub-element (j, k),
$\overline{P_{i1}}$ = center of the common limit arc of Σ_{i1} and Σ_{i2};

$$\int_{\sigma} \nu(Q)\,\partial_{n(P_i)} G(P_i, Q')\,\mathrm{d}Q' \simeq \sum_{j=1}^{80} \nu_j \left\{ \sum_{l=1}^{32} \partial_{m(P_i)} G(P_i, Q'_{jl})\,\sigma_{jl} \right\},$$

σ_{jl} $(j = 1, 2, \ldots, 32,\ l = 1, 2, \ldots, 80)$ = areas of the 80 sub-elements of the 32 annular elements of σ,
Q'_{jl} $(j = 1, 2, \ldots, 32,\ l = 1, 2, \ldots, 80)$ = center of the sub-element (j, l).

Similar approximations are made for the second of equations (58).

In this example too, the eigenfrequencies of the domain and the forced acoustical field have been computed; comparison is made between numerical and experimental results. Table 2 show the difference between the computed eigenfrequencies and the measured ones: the agreement is good (the error is about 2%). Figure 4 shows the experimental sound field curves for $ka = 5$, 10 and 20 (full lines) and the numerical predictions (dotted lines); despite the complexity of the problem, the numerical approach appears to be efficient enough for physical purposes.

TABLE 2
Eigenfrequencies of the ellipsoidal room containing a sphere; ν_c = computed values, ν_m = measured values

ν_c (Hz)	ν_m (Hz)
111·0	113
201·2	204
277·1	279
364·9	364
438·5	441
642·1	640
782·6	784
849·2	854
912·6	907
931·3	933
996·1	990
1040·6	1033

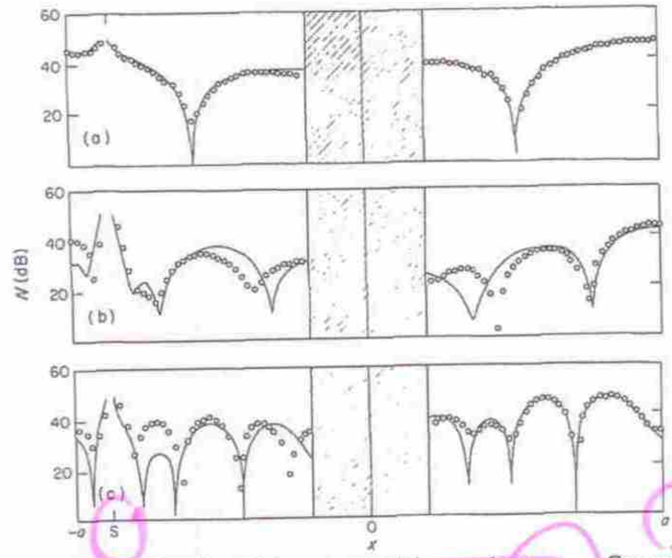


Figure 4. Sound field within the ellipsoidal room containing a sphere. ......, Computed points; —, experimental curves. (a) $ka = 5$; (b) $ka = 10$; (c) $ka = 20$.

3.3. EXTERIOR PROBLEMS

Central to this example is the aim of showing the efficiency of the use of a simple layer–double layer potential combination with complex coefficients. The present results are due to Bolomey and Tabbara [18, 19]. The diffraction of an incident plane wave by a perfectly soft cylindrical obstacle (Dirichlet problem) of radius a is dealt with. The numerical results are compared with the analytical series solution obtained in cylindrical co-ordinates (by separation of variables).

First, the Green formula is used, giving a simple layer potential description of the diffracted field:

$$\varphi(M) = \varphi_0(M) - \int_\Gamma \partial_n \varphi(P')\, G(M,P')\, \mathrm{d}P'. \tag{59}$$

The corresponding integral equation is

$$\frac{\partial_n \varphi(P)}{2} + \int_\Gamma \partial_n \varphi(P')\, \partial_{n(P)} G(P,P')\, \mathrm{d}P' = \partial_{n(P)} \varphi_0(P), \qquad \forall\, P \in \Gamma. \tag{60}$$

The wavenumber k is chosen equal to an eigenwavenumber of the interior Dirichlet problem. Equation (60) is solved numerically under the restrictive condition that the expression (59) is zero within the cylindrical obstacle (this method was proposed by Schenck [20]).

Second, the diffracted field is described by a simple layer–double layer potential combination:

$$\varphi(M) = \varphi_0(M) + \int_\Gamma \mu(P')\,[\partial_{n(P')} G(M,P') - \mathrm{i}G(M,P')]\, \mathrm{d}P'. \tag{61}$$

Here the corresponding integral equation is

$$-\frac{\mu(P)}{2} + \int_\Gamma \mu(P')\,[\partial_{n(P')} G(P,P') - \mathrm{i}G(P,P')]\, \mathrm{d}P' = -\varphi_0(P), \qquad \forall\, P \in \Gamma, \tag{62}$$

and this is solved numerically in the way proposed in section 3.1.

In Figure 5, the moduli of
equations (60) and (62) are cc
increases, the system derived fr
equation (62) is always regular
eigenwavenumber for the Diric
system produced from equation
condition that expression (59) is
numerical result, conversely, th
agreement with those from the
be shown on the drawing.

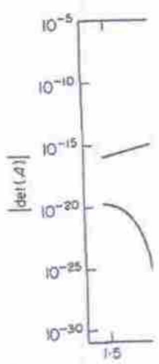


Figure 5. Comparison of the mod
(60) and (62) for, respectively, a simp

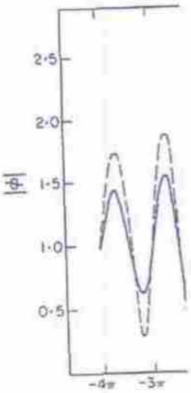
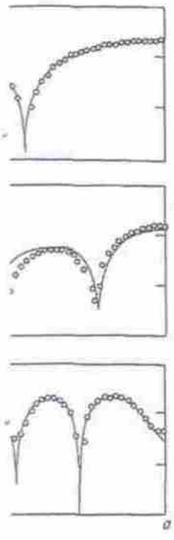


Figure 6. Comparison of the simp
cylindrical obstacle.

3.4. DIFFRACTION BY AN INFINI

It was shown in section 2.3 th
a perfectly reflecting infinitely
integral equation involves a fin
such an integral equation can



:re. ......, Computed points; —,

y of the use of a simple layer–
The present results are due to
plane wave by a perfectly soft
/ith. The numerical results are
cylindrical co-ordinates (by

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) dP'. (59)

P), $\forall P \in \Gamma$. (60)

the interior Dirichlet problem.
ition that the expression (59) is
d by Schenck [20]).
ouble layer potential combina-

$G(M, P')$] dP'. (61)

$\rho_0(P)$, $\forall P \in \Gamma$, (62)

3.1.

In Figure 5, the moduli of the determinants of the algebraic systems approximating equations (60) and (62) are compared: it appears immediately that, as the wavenumber increases, the system derived from equation (60) can be singular, while that deriving from equation (62) is always regular. The total field is computed for $k = 3{\cdot}8317/a$, which is an eigenwavenumber for the Dirichlet interior problem: for such a k, the determinant of the system produced from equation (60) is singular. Figure 6 shows that, despite the use of the condition that expression (59) is zero within the obstacle, this description provides a very bad numerical result, conversely, the representation (61) provides numerical results in very good agreement with those from the series expansion of the total field; the errors are too small to be shown on the drawing.

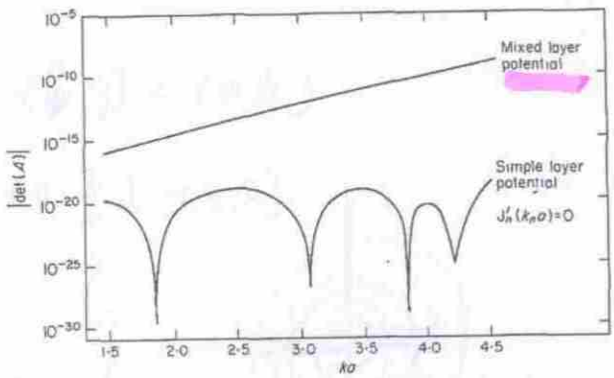

Figure 5. Comparison of the moduli of the determinants of the algebraic systems approximating equations (60) and (62) for, respectively, a simple layer potential and a mixed layer potential.

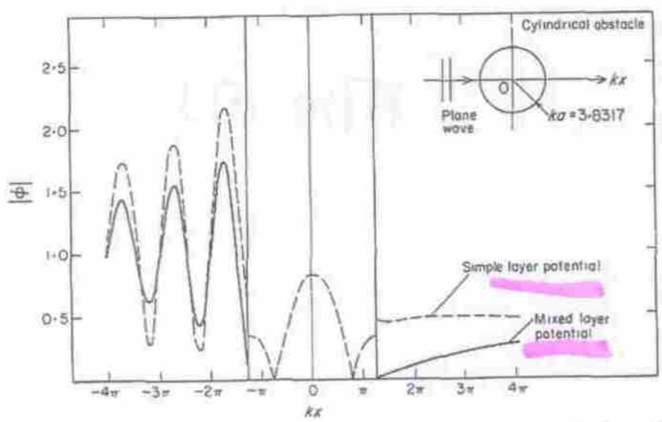

Figure 6. Comparison of the simple layer (------) and the mixed layer (——) approximate field of the soft cylindrical obstacle.

3.4. DIFFRACTION BY AN INFINITELY THIN PERFECTLY REFLECTING SCREEN

It was shown in section 2.3 that the only layer potential description of the field diffracted by a perfectly reflecting infinitely thin screen is that of a double layer one; the corresponding integral equation involves a finite part in the Hadamard sense. Hence it is useful to show how such an integral equation can be numerically approximated and solved (see also reference

[21]). Two examples are detailed here: first, the diffraction of a plane incident wave by an infinite plane strip is considered, and numerical results are compared to the Mathieu function series expansion solution; in the second example, the acoustical field due to a spherical incident wave and diffracted by a rectangular screen is numerically computed and compared to experimental results.

3.4.1. *Diffraction of a plane incident wave by an infinite plane strip*

Let an infinite plane strip be located in the $Y=0$ plane, and extended from $X=-a$ to $X=+a$. A plane incident wave, $\varphi_0=e^{ikY}$, is diffracted by the strip which is assumed to be

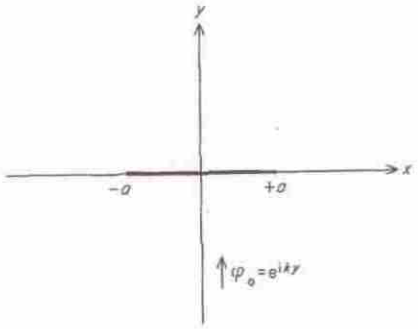

Figure 7. Geometry of the infinite plane strip.

perfectly reflecting. The diffracted field is described by a double layer potential, the density of which is the solution of the following integral equation:

$$\text{Pf.}\int_{-a}^{+a}\mu(X')\frac{\partial^2 G(X,X')}{\partial y\,\partial y'}\,dX'=-ik,\qquad -a\leqslant X\leqslant +a,$$

$$\frac{\partial^2 G(X,X')}{\partial Y\,\partial Y'}=\frac{\partial^2}{\partial Y\,\partial Y'}\left\{-\frac{i}{4}H_0[k\sqrt{(X-X')^2+(Y-Y')^2}]\right\}_{Y=Y'=0}. \qquad (63)$$

In equation (63), the finite part is defined as the limit

$$-\lim_{Y\to 0}\frac{i}{4}\int_{-a}^{+a}\mu(X')\,\frac{\partial^2}{\partial Y\,\partial Y'}\{H_0[k\sqrt{(X-X')^2+(Y-Y')^2}]\}_{Y'=0}\,dX'. \qquad (64)$$

Because of the geometrical symmetry of the data, the function $\mu(X')$ is symmetrical with respect to X'.

Upon dividing the interval $(0,+a)$ into N equal elements, the extremities of which are $[(j-1)a/N, ja/N; j=1,2,\ldots,N]$, and approximating μ by a constant μ_j on each element, the following approximation for equation (63) is obtained:

$$\sum_{j=1}^{N}\mu_j\left\{\int_{-(j-1)a/N}^{-ja/N}\frac{\partial^2 G[(n-\frac{1}{2})a/N,X']}{\partial Y\,\partial Y'}\,dX'+\text{Pf.}\int_{(j-1)a/N}^{ja/N}\frac{\partial^2 G[(n-\frac{1}{2})a/N,X']}{\partial Y\,\partial Y'}\,dX'\right\}=-ik,$$
$$n=1,2,\ldots,N. \qquad (65)$$

In this expression, the symbol Pf. is relevant only for $j=n$.

The integrals for $X'<0$ or $(X'$:
by using the series representatio
position of cylindrical waves cente

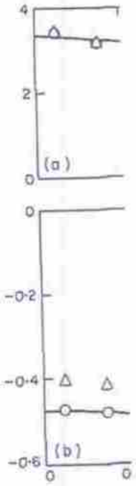

Figure 8. Diffraction by a strip, $ka=$
equations.

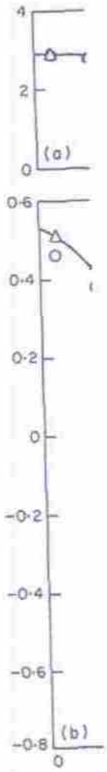

Figure 9. Diffraction by a strip, $ka=$
equations.

of a plane incident wave by an
mpared to the Mathieu function
ustical field due to a spherical
rically computed and compared

strip

and extended from $X = -a$ to
ıe strip which is assumed to be

$\longrightarrow x$

e strip.

uble layer potential, the density

$a \leqslant X \leqslant +a,$

$\overline{(Y-Y')^2]}\Big\}_{Y=Y'=0}$. (63)

$\overline{' - Y')^2]}\}_{Y'=0}\, dX'$. (64)

:tion $\mu(X')$ is symmetrical with

s, the extremities of which are
constant μ_j on each element, the

$\dfrac{-\frac{1}{2})a/N, X']}{\partial Y \partial Y'} dX' \Big\} = -ik,$

$n = 1, 2, \ldots, N.$ (65)

The integrals for $X' < 0$ or $(X' > 0, j \neq n)$ are of the Riemann type; they can be evaluated by using the series representation of $\partial'_Y\, H_0[k\sqrt{(X_n - X')^2 + (Y - Y')^2}]_{Y'=0}$ as the superposition of cylindrical waves centered at $P_j(X_j = (j - 1/2)a/N)$; differentiating this series with

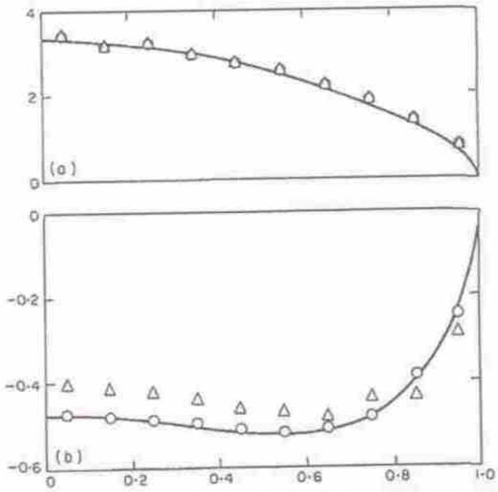

Figure 8. Diffraction by a strip, $ka = 2$. (a) Real part of μ; (b) imaginary part of μ. △, 10 equations; ○, 100 equations.

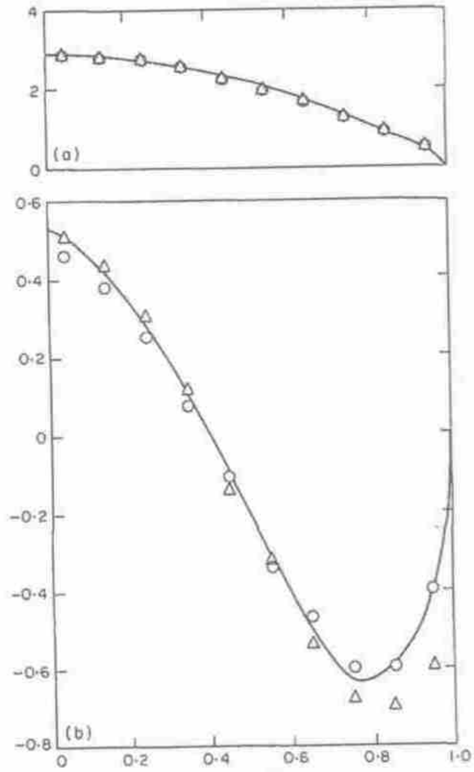

Figure 9. Diffraction by a strip, $ka = 3$. (a) Real part of μ; (b) imaginary part of μ. △, 10 equations; ○, 100 equations.

respect to Y, and letting $Y=0$, one obtains a series of Bessel functions which is next integrated term by term; in this resulting series, the terms up to the third are taken into account, the others being neglected. To evaluate the finite part, a slightly different scheme is used: the potential at a point M, due to a unit double layer supported by the nth segment, is calculated by using the asymptotic series representation of $H_0(kr)$ for $kr \ll 1$; the normal derivative of the result with respect to the normal at P_n is calculated, and, hence, the limit for $M \to P_n$ is taken.

Figures 8 and 9 show the exact value of μ for $ka=2$ and $ka=3$, obtained by a series representation with Mathieu functions. The numerical approximations corresponding to $N=10$ and $N=100$ are very consistent with the exact values; it is to be remarked that the accuracy obtained increases from $N=10$ to $N=100$, which proves experimentally the convergence of the procedure; but the increase in accuracy is small compared to the computation time increase.

3.4.2. *Diffraction of a spherical incident wave by a plane rectangular reflecting screen*

A plane rectangular screen Σ, the dimensions of which are $2a$ and $2b$, lies in the $Z=0$ plane, as shown in Figure 10; a spherical point source S is located on the Z axis. The diffracted field is described by a double layer potential, the density of which is the solution of

$$\text{Pf.}\int_{\Sigma} \mu(P')\,\frac{\partial^2 G(P,P')}{\partial n(P)\,\partial n(P')}\,\mathrm{d}P' = -\partial_{n(P)}\varphi_0(P), \qquad \forall\, P \in \Sigma, \tag{66}$$

with

$$G(M,M') = -\frac{e^{ikr(M,M')}}{4\pi r(M,M')}, \qquad \varphi_0(M) = -\frac{e^{ikr(S,M)}}{4\pi r(S,M)}.$$

To solve equation (66), use has been made of the symmetry properties of the function $\mu(P') = \mu(X,Y)$:

$$\mu(X,Y) = \mu(\pm X, \pm Y).$$

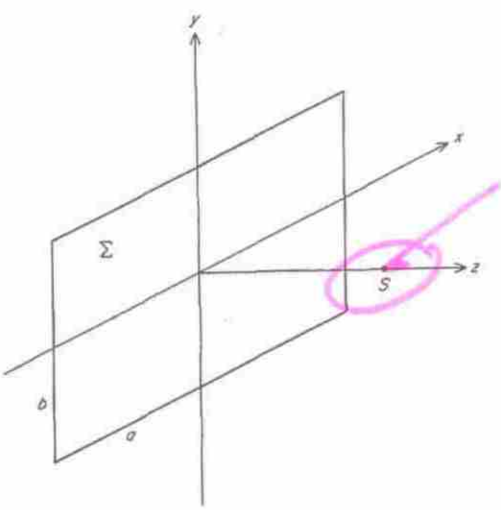

Figure 10. Geometry of the rectangular screen.

For the numerical approximatio[n]
equal elements, the centers of whi[ch]

$X_r = (r - m -$

and which lie within the lines

$X = X_r$:

The parameters m and n are integ[ers]
$2m$) and $(1, 2, \ldots, 2n)$, respective[ly]
computation, it is useful to denot[e]
used the following index: $i = (s -$
$2mn$, and $i = (s-1)m + r + (2n -$
$2mn+2, \ldots, 4mn$. With this index,

$X_i = (r - m -$

with s = integer part of $(1 + (i-1$
part of $(1 - 2n + (i-1)/m)$ and $r =$
A_{ij} of the linear system approxim[ating]

A_{ij}

To obtain expression (67), the el[ements]
centered at points denoted M_{ji}. T[he]
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For the numerical approximation of equation (66), the screen is divided into rectangular equal elements, the centers of which are

$$X_r = (r - m - 0{\cdot}5)\,a/m, \qquad Y_s = (s - n - 0{\cdot}5)\,b/n,$$

and which lie within the lines

$$X = X_r \pm 0{\cdot}5a/m, \qquad Y = Y_s \pm 0{\cdot}5b/n.$$

The parameters m and n are integers; the integers r and s take the successive values $(1, 2, \ldots, 2m)$ and $(1, 2, \ldots, 2n)$, respectively. The number of elements so defined is $4mn$. For easier computation, it is useful to denote the elements by one index only, to do so, the author has used the following index: $i = (s-1)m + r$ $(s = 1, 2, \ldots, 2n;\ 1 \leqslant r \leqslant m)$, so that $i = 1, 2, \ldots, 2mn$, and $i = (s-1)m + r + (2n-1)m$ $(s = 1, 2, \ldots, 2n;\ m+1 \leqslant r \leqslant 2m)$ so that $i = 2mn+1, 2mn+2, \ldots, 4mn$. With this index, the co-ordinates of the center M_i of an element are given by

$$X_i = (r - m - 0{\cdot}5)\,a/m, \qquad Y_i = (s - n - 0{\cdot}5)\,b/n,$$

with s = integer part of $(1 + (i-1)/m)$ and $r = i - (s-1)m$ for $1 \leqslant i \leqslant 2mn$, and s = integer part of $(1 - 2n + (i-1)/m)$ and $r = i - mn - (s-1)m$ for $2mn + 1 \leqslant i \leqslant 4mn$. The coefficients A_{IJ} of the linear system approximating equation (66) are the following:

$$A_{Ij \neq I} = \sum_{l=1}^{9} \sigma_{Jl} \left. \frac{\partial^2 G(M_I, M_{Jl})}{\partial Z\, \partial Z'} \right|_{Z = Z' = 0}. \tag{67}$$

To obtain expression (67), the element j is divided into 9 equal rectangular sub-elements centered at points denoted M_{Jl}. The diagonal coefficients A_{II} are obtained by calculating the finite part of the integral of $\partial^2 G/\partial Z \partial Z'$ on a disk, the area of which is equal to that of the rectangular element. Finally equation (66) is replaced by a linear system which takes into account the symmetry properties of the μ_i induced by those of the function $\mu(X')$.

The diffracted field is approximated in a very simple way by

$$\varphi_d(M) \simeq \sum_{i=1}^{4mn} \mu_i \frac{ab}{mn} \left. \frac{\partial G(M, M_i)}{\partial Z'} \right|_{Z' = 0},$$

where M_i is the point with co-ordinates (X_i, Y_i, Z').

In Figures 11(a–c) the experimental curves (in dB levels) are given for different values of ka; the ratio a/b is equal to 3/2, the source–screen distance is $2a$. It can be seen that the accuracy of the computed points is good enough to provide a satisfactory description of the phenomenon.

Remark. In the book by Delves and Walsh [22], a regularization method is proposed. It consists in multiplying the non-integrable kernel $\partial_{n_0} \partial_n G(P_0, P)$ by a regularizing one $\bar{K}(P_0', P_0)$ and integrating over Γ to obtain a regular kernel:

$$K(P_0', P_0) = \int_\Gamma \bar{K}(P_0', P_0)\, \partial_{n(P_0)} \partial_{n(P)} G(P_0, P)\, \mathrm{d}P_0.$$

The numerical approximation of the integral equation so obtained will be a linear system, the coefficients of which are integrals over Γ! Such a method will be very expensive because of the time needed for computing each integral; furthermore, less accuracy is to be expected because the errors in the integrals' computation add to those in the evaluation of the original and the regularizing kernels.

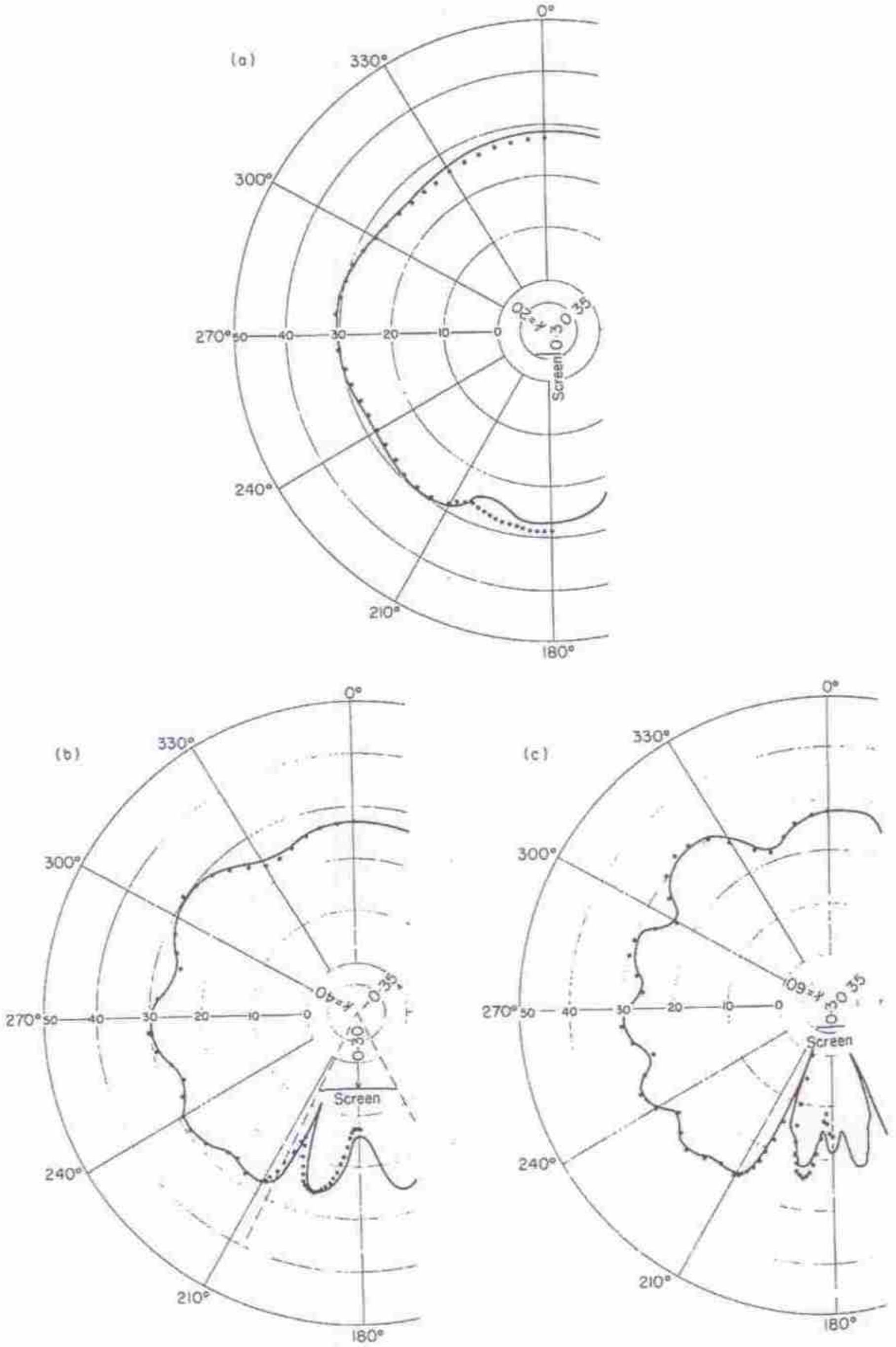

Figure 11. Diffraction by a rectangular screen. These figures show the sound level as measured (——) and calculated (……) along a circle, of radius 0·35 m, centered at the point source, and lying in the (x, z) plane. The screen has the following dimensions: $a = 0{\cdot}15$ m, $b = 0{\cdot}10$ m; the source-screen distance is 0·30 m. The wavenumbers k correspond to the following wavelengths: (a) 0·314 m; (b) 0·236 m; (c) 0·157 m.

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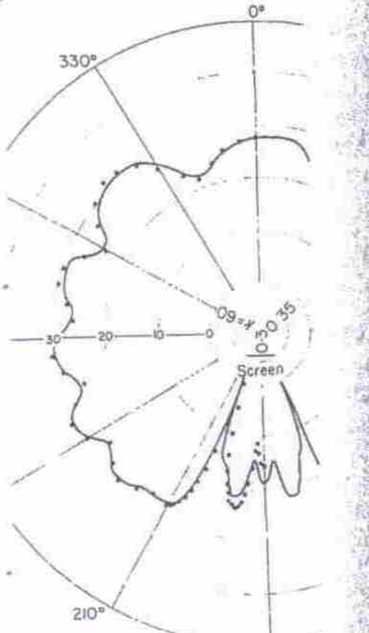

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4. CONCLUSION

The first result of this paper has been the demonstration that the solution of any boundary value problem for the scalar Helmholtz equation can always be represented by a layer potential; the layer type (simple, double, or a combination of simple and double layer) can be chosen in such a way that the solution of the integral equation so obtained is subject to the same existence and uniqueness conditions as those of the partial differential system.

The second result is that simple approximations can be made, leading to a convergent numerical procedure for solving the derived integral equations (the convergence is not established here, but is shown by numerical experiments). Particularly, even when a finite part of an integral occurs (non-integrable kernels), the approximation procedure remains efficient; it appears that the more complicated ideas proposed by several authors to avoid the difficulty of a non-integrable kernel are not useful or necessary.

The third result is that integral equation methods are efficient within the frequency range corresponding to ratios (wavelength/obstacle dimensions) of about 1/20 to 20. For such a frequency range, the long wavelength and the short wavelength approximations break down.

One can note further that such methods are powerful for many other boundary value problems of mathematical physics, for example in electromagnetism, hydrodynamics, statical and dynamical elasticity, thin plate vibration problems, etc. In the complementary bibliography, the more recent important works about integral equations in mechanics are listed.

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