海洋大學河海工程學系九十四學年度 第一學期 ■期中 □小考 考試命題紙

考 試 科 目	開課系級	考試日期	印製份數	答案紙	命題教師	備 註
工程數學一	二 A, B	10 月 24 日	111	■ 需 □不需	陳桂鴻 呂學育	第一次大考

1. A Bernoulli equation of $xy' + y = \dfrac{1}{y}$,

(a) Linear or nonlinear (2%), why? (2%)

(b) Exact (Yes or No) (2%), why? (2%)

(c) Solve by Separable variable method. (6%)

(d) Solve by Linear O.D.E method (Convert the O.D.E to a linear equation by using the change of variable method). (7%)

(e) Solve by the Exact O.D.E method. (9%)

Ans: (a) nonlinear, 有因變數的二次項.

(b) nonexact, $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$.

(c) $\dfrac{dy}{dx} = (\dfrac{1-y^2}{y})\dfrac{1}{x}$, $\int \dfrac{y}{1-y^2}dy = \int \dfrac{1}{x}dx$, $\dfrac{-1}{2}\ln|1-y^2| = \ln|x| + c$, $(1-y^2)^{-\frac{1}{2}} = x + c_1$.

(d) $yy' + y^2 = \dfrac{1}{x}$, **Let** $u = y^2$, $u' = 2yy'$, $\therefore \dfrac{1}{2}u' + u = \dfrac{1}{x}$, $-\dfrac{1}{2}\ln|1-u| = \ln|x| + c$, $(1-y^2)^{-\frac{1}{2}} = x + c_1$.

(e) $(y^2-1)dx + xydy = 0$, $\dfrac{\partial \mu M}{\partial y} = \dfrac{\partial \mu N}{\partial x}$, $(y^2-1)\dfrac{\partial \mu}{\partial y} + 2\mu y = xy\dfrac{\partial \mu}{\partial x} + \mu y$; **Let** $\mu = \mu(x)$, $\therefore \mu = x$

$\dfrac{\partial \phi}{\partial x} = \mu M$, $\therefore \phi = \dfrac{1}{2}x^2y^2 - \dfrac{1}{2}x^2 + h(y)$; $\dfrac{\partial \phi}{\partial y} = \mu N$, $x^2y + h'(y) = x^2y$, $\therefore h(y) = c$

$\therefore \dfrac{1}{2}x^2y^2 - \dfrac{1}{2}x^2 + c = 0$.

2.The Ricatti equation $y' = y^2 - \dfrac{1}{x}y - \dfrac{4}{x^2}$ **by using the solution** $y_2 = y_1 + \dfrac{1}{z}$ **with** $y_1 = 2/x$, **we obtain** $y_2 = \dfrac{2}{x} + \dfrac{1}{-\frac{1}{4}x + Cx^{-3}}$, $C \in \Re$. **By setting** $C = 0$, **we have** $y_2 = -\dfrac{2}{x}$, **solve** $y_3 = -\dfrac{2}{x} + \dfrac{1}{z}$, **please find** y_3. **(10%)**

Ans: $y_3 = -\dfrac{2}{x} + \dfrac{1}{z}$, $y_3' = \dfrac{2}{x^2} - \dfrac{1}{z^2}z'$, $\dfrac{2}{x^2} - \dfrac{1}{z^2}z' = (\dfrac{-2}{x} + \dfrac{1}{z})^2 - \dfrac{1}{x}(\dfrac{-2}{x} + \dfrac{1}{z}) - \dfrac{4}{x^2}$, $\therefore z' - \dfrac{5}{x}z = -1$

$I = x^{-5}$, $x^{-5}z' - 5x^{-6}z = -x^{-5}$, $x^{-5}z = \dfrac{1}{4}x^{-4} + c$, $\therefore z = \dfrac{x}{4} + cx^5$, $y_3 = \dfrac{-2}{x} + \dfrac{1}{\frac{x}{4} + cx^5}$.

3. Solve the singular solution and general solution of the Clairauts equation $y = x\frac{dy}{dx} + f(\frac{dy}{dx})$**, where** $f(\frac{dy}{dx}) = -e^{2y'}$**. (10%)**

Ans: $y = xy' - e^{2y'}$**, Let** $p = y'$, $\therefore y = xp - e^{2p}$, $y' = p + xp' - 2p'e^{2p}$, $\therefore p'(x - 2e^{2p}) = 0$

general solution: $\begin{cases} p' = 0 \\ y = xp - e^{2p} \end{cases}$, $p = c$, $\therefore y = cx - e^{2c}$

singular solution: $\begin{cases} x - 2e^{2p} = 0 \\ y = xp - e^{2p} \end{cases}$, $p = \frac{1}{2}\ln\left|\frac{x}{2}\right|$, $\therefore y = \frac{1}{2}x\ln\left|\frac{x}{2}\right| - \frac{x}{2}$.

4. (a) is the differential equation $(1-x)y' - 4x\sin(y) = \cos(x)$ **linear or nonlinear in** y**? (3%)**

Ans: nonlinear.

(b) is the differential equation $(y^2 - 1)dx = xdy$ **linear or nonlinear in** x**? (3%)**

Ans: linear.

(c) solve the separable differential equation $\frac{dy}{dx} + \frac{y}{x} = 0, \ y(1) = 1$

Ans: $\frac{dy}{y} = -\frac{dx}{x}$

$\ln y = \ln x + \ln c \rightarrow y = cx^{-1}$

$y(0) = 0 \rightarrow y = x^{-1}$.

(d) What is the slope of the tangent line to the graph of the solution $y' = 6\sqrt{y} + 5x^3$ **that through (-1, 1) ? (4%)**

Ans: The slope of the tangent line ： $y' = 6\sqrt{1} + 5(-1)^3 = 1$.

(e) Match the given differential equations with one or more of the solutions

(a) $y = 0$**, (b)** $y = 2$**, (c)** $y = 2 + 2x^2$**, (d)** $y = 2x^2$**, (e)** $y = -2x^2$ **(5%)**

$x\frac{dy}{dx} = 2y$ ； $\frac{dy}{dx} = 2y - 4$

Ans: $x\frac{dy}{dx} = 2y \ \Rightarrow (a), (d), (e)$, $\frac{dy}{dx} = 2y - 4 \ \Rightarrow (b), (c)$.

5. Solve the initial-value problem $\frac{dy}{dx} = (3x - y)^2 + 6x - 2y$, $y(0) = -3$ **(15%)**

Ans: Let $u = 3x - y \ \rightarrow \ \frac{dy}{dx} = 3 - \frac{du}{dx}$

$$\frac{du}{dx} = -u^2 - 2u + 3 = -(u+3)(u-1)$$

$$\frac{du}{-(u+3)(u-1)} = dx$$

$$\frac{1}{4}\int (\frac{1}{u+3} - \frac{1}{u-1})du = \int dx$$

$$\ln(\frac{u+3}{u-1})^{\frac{1}{4}} = x + c$$

$$e^{4x+c} = \frac{u+3}{u-1} \quad \rightarrow \quad e^{4x+c} = \frac{3x-y+3}{3x-y-1}$$

$$y(0) = -3 \quad \rightarrow \quad c = \ln 3.$$

The solution of the initial-value problem is $3e^{4x} = \dfrac{3x-y+3}{3x-y-1}$.

6. $(2y\sin x - 3)dx - \cos x dy = 0$

(a) Is it exact? (2%) Why? (3%)

Ans: It is not exact.

$$\frac{\partial M}{\partial y} = 2\sin x \neq \frac{\partial N}{\partial x} = \sin x \quad .$$

(b) Solve it by using an integrating factor. (10%)

Ans: Let $f(x) = \dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = -\tan x$

$$\rightarrow \quad \mu(x) = \int e^{f(x)dx} = \cos x$$

$$(2y\sin x\cos x - 3\cos x)dx - \cos^2 x dy = 0$$

$$\int (2y\sin x\cos x - 3\cos x)dx = -\frac{1}{2}y\cos 2x - 3\sin x + f(y)$$

$$\int -\cos^2 x dy = -\frac{1}{2}y\cos 2x - \frac{y}{2} + g(x)$$

$$\rightarrow \quad \frac{1}{2}y\cos 2x + 3\sin x + \frac{y}{2} = c\,.$$