海洋大學河海工程學系九十四學年度 第一學期 ■期中 □小考 考試命題紙

考試科目	開課系級	考試日期	印製份數	答案紙	命題教師	備註
工程數學(一)	二 A,B	11 月 18 日	111	■需 □不需	陳桂鴻 呂學育	第二次期中考 共 1 頁

1. (a) Determine the complementary solution, the particular solution and general solution of a nonhomogeneous linear DE $y^{(4)}+y'''=1-x^2e^{-x}$. (10%)

Ans: $\lambda^4+\lambda^3=0$, $\lambda=-1,0,0,0$, $y_c=c_1e^{-x}+c_2+c_3x+c_4x^2$

Let $y_p=ax^3+(bx^3+cx^2+dx)e^{-x}$

$$y_p'''=6a+6be^{-x}-3(6bx+2c)e^{-x}+3(3bx^2+2cx+d)e^{-x}-(bx^3+cx^2+dx)e^{-x}$$

$$y_p^{(4)}=-24be^{-x}+6(6bx+2c)e^{-x}-4(3bx^2+2cx+d)e^{-x}+(bx^3+cx^2+dx)e^{-x}$$

$$a=\frac{1}{6},\ b=\frac{1}{3},\ c=3,\ d=12,\ \therefore y_p=\frac{1}{6}x^3+(\frac{1}{3}x^3+3x^2+12x)e^{-x}$$

$$y=c_1e^{-x}+c_2+c_3x+c_4x^2+\frac{1}{6}x^3+(\frac{1}{3}x^3+3x^2+12x)e^{-x}$$

(b) Find the general solution of $x^4y''+x^3y'+4x^2y=1$, given that $y_1(x)=x^2$ is a solution of the associated homogeneous equation. (5%)

Ans: $y''+\frac{1}{x}y'+\frac{4}{x^2}y=\frac{1}{x^4}$, $p(x)=\frac{1}{x}$, $e^{-\int p(x)dx}=x^{-1}$, $y_2=y_1\int\frac{e^{-\int p(x)dx}}{y_1^2}dx=x^2\int x^{-5}dx=\frac{-1}{4}x^{-2}$

$$\therefore y_c=c_1x^2-c_2\frac{1}{4}x^{-2}$$

Let $y_p=u_1x^2+u_2\frac{1}{4}x^{-2}$, $\omega=-x^{-1}$, $u_1=\frac{-1}{16}x^{-4}$, $u_2=-\ln|x|$, $\therefore y_p=\frac{-1}{16}x^{-2}-\frac{1}{4}x^{-2}\ln|x|$

$$\therefore y=c_1x^2-c_2\frac{1}{4}x^{-2}-\frac{1}{16}x^{-2}-\frac{1}{4}x^{-2}\ln|x|=c_1x^2-c_3x^{-2}-\frac{1}{4}x^{-2}\ln|x|$$

2. Cauchy-Euler equation $x^2y''+10xy'+8y=x^2$. Find the complementary solution by using

(a) Change of variable method ($t=\ln x$). (10%)

Ans: Let $t=\ln x$, $x=e^t$

$$\therefore x^2y''+10xy'+8y=x^2\rightarrow Y''(t)+9Y'(t)+8Y(t)=e^{2t}$$

$$\lambda^2+9\lambda+8=0,\ \lambda=-8,-1,\ \therefore Y_c=c_1e^{-8t}+c_2e^{-t}$$

Let $Y_p=u_1e^{-8t}+u_2e^{-t}$, $w=7e^{-9t}$

$$u_1'=\frac{\begin{vmatrix}0 & e^{-t}\\ e^{2t} & -e^{-t}\end{vmatrix}}{7e^{-9t}}=\frac{-1}{7}e^{10t},\ u_1=\int u_1'dt=\frac{-1}{70}e^{10t}$$

$$u_2' = \frac{\begin{vmatrix} e^{-8t} & 0 \\ -8e^{-8t} & e^{2t} \end{vmatrix}}{7e^{-9t}} = \frac{1}{7}e^{3t}, \quad u_2 = \int u_2' dt = \frac{1}{21}e^{3t}$$

$$Y_p = \frac{-1}{70}e^{10t}e^{-8t} + \frac{1}{21}e^{3t}e^{-t} = \frac{-1}{70}e^{2t} + \frac{1}{21}e^{2t} = \frac{1}{30}e^{2t}, \quad \therefore y_p = \frac{1}{30}x^2$$

(b) Let $y = x^m$ and looks for m. (5%)

Ans: $m^2 + 9m + 8 = 0$, $m = -8, -1$, $y_c = c_1 x^{-8} + c_2 x^{-1}$

Let $y_p = u_1 x^{-8} + u_2 x^{-1}$, $w = 7x^{-10}$

$$u_1' = \frac{\begin{vmatrix} 0 & x^{-1} \\ 1 & -x^{-2} \end{vmatrix}}{7x^{-10}} = \frac{-1}{7}x^9, \quad u_1 = \int u_1' dt = \frac{-1}{70}x^{10}$$

$$u_2' = \frac{\begin{vmatrix} x^{-8} & 0 \\ -8x^{-9} & 1 \end{vmatrix}}{7x^{-10}} = \frac{1}{7}x^2, \quad u_2 = \int u_2' dt = \frac{1}{21}x^3$$

$$y_p = \frac{-1}{70}x^{10}x^{-8} + \frac{1}{21}x^3x^{-1} = \frac{-1}{70}x^2 + \frac{1}{21}x^2 = \frac{1}{30}x^2$$

(c) If we has found one solution, $y_1(x) = x^{-1}$, find y_2 by using reduction order method. (10%)

Ans: $x^2 y'' + 10xy' + 8y = x^2 \rightarrow y'' + \frac{10}{x}y' + \frac{8}{x^2}y = 1$

$$p(x) = \frac{10}{x}, \quad e^{-\int p(x)dx} = x^{-10}, \quad y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2}dx = x^{-1}\int x^{-8}dx = \frac{-1}{7}x^{-8}$$

3. Solve the singular solution and general solution of the Clairauts equation $y = x\frac{dy}{dx} + f(\frac{dy}{dx})$, where $f(\frac{dy}{dx}) = -e^{2y'}$. (10%)

Ans: $y = xy' - e^{2y'}$, **Let** $p = y'$, $\therefore y = xp - e^{2p}$, $y' = p + xp' - 2p'e^{2p}$, $\therefore p'(x - 2e^{2p}) = 0$

general solution: $\begin{cases} p' = 0 \\ y = xp - e^{2p} \end{cases}$, $p = c$, $\therefore y = cx - e^{2c}$

singular solution: $\begin{cases} x - 2e^{2p} = 0 \\ y = xp - e^{2p} \end{cases}$, $p = \frac{1}{2}\ln\left|\frac{x}{2}\right|$, $\therefore y = \frac{1}{2}x\ln\left|\frac{x}{2}\right| - \frac{x}{2}$.

4. Consider (a), (b), (c), (d), determine whether the given set of function is linearly dependent or linearly independent on the interval $(0, \infty)$.

(a) $f_1(x) = \cos(2x)$, $f_2(x) = \sin(2x)$ **(3%)**

Ans: $W(f_1, f_2) = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = 2\cos^2(2x) + 2\sin^2(2x) = 2 \neq 0 \quad \therefore \textit{linearly independent}$

(b) $f_1(x)=e^{2x},\ \ f_2(x)=e^{-2x}$ **(3%)**

Ans: $W(f_1,f_2)=\begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}=-2-2=-4\neq 0\ \ \therefore\ linearly\ independent$

(c) $f_1(x)=x^2,\ \ f_2(x)=x+1$ **(3%)**

Ans: $W(f_1,f_2)=\begin{vmatrix} x^2 & x+1 \\ 2x & 1 \end{vmatrix}=-x^2-2x\neq 0\ \ \therefore\ linearly\ independent$

(d) $f_1(x)=1+x,\ \ f_2(x)=-x,\ \ f_3(x)=-x^2$ **(3%)**

Ans: $W(f_1,f_2,f_3)=\begin{vmatrix} 1+x & -x & -x^2 \\ 1 & -1 & -2x \\ 0 & 0 & -2 \end{vmatrix}=2(1+x)-2x=2\neq 0\ \ \therefore\ linearly\ \ independent$

(e) is the differential equation $(y^2-1)dx=xdy$ **linear or nonlinear in** x **? why ? (3%)**

Ans: $(y^2-1)x'=x\ \ \ linear\ \ \ \because$ 無互乘項 無非線性函數

5. Given second-order differential equations on the interval $(-\infty,0)$

(1) $x^2y''-2xy'+2y=5x^3\cos 3x$

(a) solve the associated homogeneous equation (y_c **) (4%)**

Ans: Let $y=x^m\ ,\ y'=mx^{m-1}\ ,\ y''=m(m-1)x^{m-2}$

$$\rightarrow m(m-1)x^m-2mx^m+2x^m=0\ \ \rightarrow m^2-3m+2=0\ \ \rightarrow m=1,2$$

$$\rightarrow y_c=c_1x+c_2x^2$$

(b) find the particular solution of the nonhomogeneous equation (y_p **) (6%)**

Ans: Let $y_p=\phi_1(x)x+\phi_2(x)x^2$

$$x^2y''-2xy'+2y=5x^3\cos 3x\ \ \rightarrow y''-\frac{2}{x}y'+\frac{2}{x^2}y=5x\cos 3x$$

$$W=\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}=x^2\ ,\ W_1=\begin{vmatrix} 0 & x^2 \\ 5x\cos 3x & 2x \end{vmatrix}=-5x^3\cos 3x\ ,\ W_2=\begin{vmatrix} x & 0 \\ 1 & 5x\cos 3x \end{vmatrix}=5x^2\cos 3x$$

$$\phi_1'=\frac{W_1}{W}=-\frac{5x^3\cos 3x}{x^2}=-5x\cos 3x\ \ \rightarrow \phi_1(x)=-\frac{5}{3}x\sin 3x-\frac{5}{9}\cos 3x$$

$$\phi_2'=\frac{W_2}{W}=\frac{5x^2\cos 3x}{x^2}=5\cos 3x\ \ \rightarrow \phi_2(x)=\frac{5}{3}\sin 3x$$

$$\therefore y_p=x(-\frac{5}{3}x\sin 3x-\frac{5}{9}\cos 3x)+x^2(\frac{5}{3}\sin 3x)$$

(2) $x^2y'' - 3xy' + 4y = 0,\ \ y(-1) = 2,\ \ y'(-1) = 4$. **Find the solution of the differential equation (8%).**

Ans: Let $-x = t$

$$t^2\frac{d^2y}{d^2t} - 3t\frac{dy}{dt} + 4y = 0 \quad y(1) = 2\ ,\ y'(1) = -4$$

Let $y = t^m \quad y'' = m(m-1)x^{m-2} \quad y' = mx^{m-1}$

$$\rightarrow (m^2 - mt + 4)x^t = 0 \quad \rightarrow m = 2,2 \quad \rightarrow y = c_1t^2 + c_2t^2\ln t$$

from $I.C.\quad y(1) = 2\ ,\ y'(1) = -4 \quad \rightarrow y = 2t^2 - 8t^2\ln t \quad \rightarrow y = 2x^2 - 8x^2\ln(-x)\ ,\ x < 0$

6. Solve the given initial-value problem

$$\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 4x = \cos(2t)\ ;\ x(0) = 0\ ,\ x'(0) = 0$$

(a) if $\beta = 0$ **find the solution of the differential equation (5%)**

Ans： $\beta = 0 \quad \rightarrow \frac{d^2x}{dt^2} + 4x = \cos(2t)$

1. Let $x = e^{mt}$

$$(m^2 + 4)e^{mt} = 0 \quad \rightarrow m = \pm 2i \quad \rightarrow x_h = c_1\cos 2t + c_2\sin 2t$$

2. Let $x_p = \phi_1(x)\cos 2t + \phi_2(x)\sin 2t$

$$W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\ ,\ W_1 = \begin{vmatrix} 0 & \sin 2t \\ \cos 2t & 2\cos 2t \end{vmatrix} = -\sin 2t\cos 2t\ ,\ W_2 = \begin{vmatrix} \cos 2t & 0 \\ -2\sin 2t & \cos 2t \end{vmatrix} = \cos^2 2t$$

$$\phi'_1 = \frac{W_1}{W} = \frac{-\sin 2t\cos 2t}{2} \quad \rightarrow \phi_1 = \frac{\cos 4t}{16}$$

$$\phi'_2 = \frac{W_2}{W} = \frac{\cos^2 2t}{2} \quad \rightarrow \phi_2 = \frac{t}{4} + \frac{1}{16}\sin 4t$$

$$x_p = \cos 2t(\frac{\cos 4t}{16}) + \sin 2t(\frac{t}{4} + \frac{1}{16}\sin 4t)$$

3. $x = x_h + x_p = c_1\cos(2t) + c_2\sin(2t) + \cos(2t)[\frac{\cos(4t)}{16}] + \sin(2t)[\frac{t}{4} + \frac{1}{16}\sin(4t)]$

from $I.C.\quad x(0) = 0\ ,\ x'(0) = 0 \quad \rightarrow c_1 = \frac{-1}{16} \quad c_2 = 0$

$$\rightarrow x = \frac{-1}{16}\sin(2t) + \cos(2t)[\frac{\cos(4t)}{16}] + \sin(2t)[\frac{t}{4} + \frac{1}{16}\sin(4t)]$$

$$\rightarrow x = \frac{t\sin(2t)}{4} + \frac{1}{16}\left([1 - 2\sin^2(2t)]\cos(2t) + 2\sin^2(2t)\cos(2t) - \cos(2t)]\right) \quad \rightarrow x = \frac{t\sin(2t)}{4}$$

(b) if $\beta = 5$ **find the solution of the differential equation (7%)**

Ans： $\beta = 5 \quad \frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = \cos(2t)$

1. Let $x = e^{mt}$

$$(m^2 + 5m + 4) = 0 \quad \rightarrow m = -1, -4 \quad \rightarrow x_h = c_1e^{-t} + c_2e^{-4t}$$

2. Let $x_p = A\cos 2t + B\sin 2t \;\; \rightarrow x_p' = -2A\sin 2t + 2B\cos 2t \;\; \rightarrow x_p'' = -4A\cos 2t - 4B\sin 2t$

$$\rightarrow 10B\cos 2t - 10A\sin 2t = \cos 2t \;\; \rightarrow B = \frac{1}{10} \;,\; A = 0$$

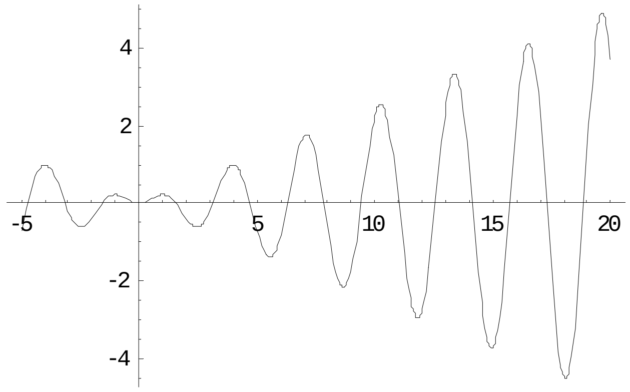
3. $x = x_h + x_p = c_1 e^{-t} + c_2 e^{-4t} + \frac{1}{10}\sin(2t)$

from $I.C.\;\; x(0) = 0 \;,\; x'(0) = 0 \;\; \rightarrow x = -\frac{1}{15}e^{-t} + \frac{1}{15}e^{-4t} + \frac{1}{10}\sin(2t)$

(c) if $t \rightarrow \infty$ **compare your solutions a) and b) (** $x \rightarrow \infty$ **?,…) (5%)**

Ans: (a) diverge

$$t \rightarrow \infty \;\; \Rightarrow \;\; \frac{t}{4} \rightarrow \infty \;\;,\;\; \sin(2t) \rightarrow oscillate$$



(b) converge

$$t \rightarrow \infty \;\; \Rightarrow \;\; -\frac{1}{15}e^{-t} \rightarrow 0 \;\;,\;\; \frac{1}{15}e^{-4t} \rightarrow 0 \;\;,\;\; \frac{1}{10}\sin(2t) \rightarrow oscillate$$