海洋大學河海工程學系九十四學年度 第一學期 ■期中 □小考 考試命題紙

考 試 科 目	開課系級	考試日期	印製份數	答案紙	命題教師	備 註
工程數學一	二 A	12 月 26 日		■ 需 □不需	陳桂鴻 呂學育	第三次大考

學生可帶 □書本 ■計算機 □其他__________ ■皆不可 共 1 頁，第 1 頁

1. A beam with flexural rigidity *EI* and length *L* is subject to the load per unit length *w*(*x*)

(1) Show that the differential equation of the deflection is $EI\dfrac{d^4y(x)}{dx^4}=w(x).$ **7%**

Ans:

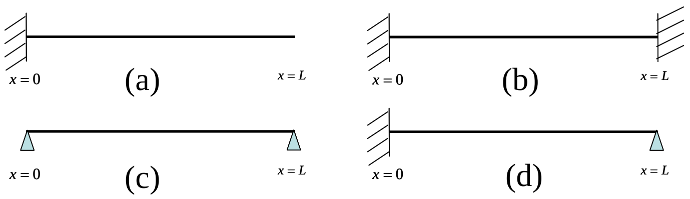
$$\frac{d^2M}{dx^2}=w(x) \qquad (1)$$

$$M(x)=EIKk \qquad (2)$$

$$k=\frac{y''}{[1+(y')^2]^{3/2}} \quad \because 微小變形 \quad \therefore y'\approx 0 \quad 故\, k=y''$$

$$\therefore M(x)=EIy'' \text{ 代入(1)得 } \frac{d^2(EIy'')}{dx^2}=w(x)\rightarrow EIy^{(4)}=w(x)$$

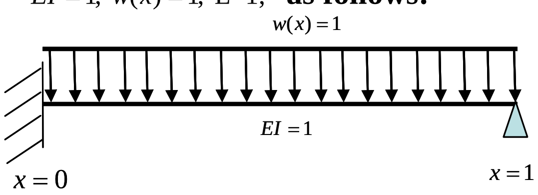
(2) Write the boundary conditions as follows 8%



Ans: (a) $y(0)=0,\ y'(0)=0,\ y''(L)=0,\ y'''(L)=0$ (b) $y(0)=0,\ y'(0)=0,\ y(L)=0,\ y'(L)=0$

(c) $y(0)=0,\ y''(0)=0,\ y(L)=0,\ y''(L)=0$ (d) $y(0)=0,\ y'(0)=0,\ y(L)=0,\ y''(L)=0$

(3) When the beam is embedded at its left end and simply supported at it right end and $EI=1,\ w(x)=1,$ L=1, **as follows:**



Find the deflection of the beam by using

(a) the method of undermined coefficients 8%

Ans: Let $y=e^{\lambda x}$ $\therefore y_c=c_1+c_2x+c_3x^2+c_4x^3$

Let $y_p=ax^4,\ y_p'=4ax^3,\ y_p''=12ax^2,\ y_p'''=24ax,\ y_p^{(4)}=24a,\ \therefore a=\dfrac{w(x)}{24EI}$

$y=c_1+c_2x+c_3x^2+c_4x^3+\dfrac{w(x)}{24EI}x^4$ and B.C. are $y(0)=0,\ y'(0)=0,\ y(L)=0,\ y''(L)=0$

$$y(0)=0\rightarrow c_1=0\,,\quad y'(0)=0\rightarrow c_2=0\,,$$

$$y(L)=0\ \text{ and }\ y''(L)=0\rightarrow c_3=\frac{3w(x)}{48EI}L^2\,, c_4=\frac{-5w(x)}{48EI}L$$

$$\therefore y=\frac{1}{16}x^2-\frac{5}{48}x^3+\frac{1}{24}x^4$$

(b) Tayler series expansion method 10%

Ans: $y(x)=y(0)+\frac{y'(0)}{1!}x+\frac{y''(0)}{2!}x^2+\frac{y'''(0)}{3!}x^3+\frac{y^{(4)}(0)}{4!}x^4+\frac{y^{(5)}(0)}{5!}x^5\cdots$

Let $y(0)=a\,,\ y'(0)=b\,,\ y''(0)=c\,,\ y'''(0)=d$

$$EIy^{(4)}=w(x)\rightarrow y^{(4)}(0)=\frac{w(x)}{EI}\,,\quad EIy^{(5)}=0\rightarrow y^{(5)}(0)=0$$

$$y(x)=a+bx+\frac{c}{2}x^2+\frac{d}{6}x^3+\frac{w(x)}{24EI}x^4,\quad y'(x)=b+cx+\frac{d}{2}x^2+\frac{w(x)}{6EI}x^3,\quad y''(x)=c+dx+\frac{w(x)}{2EI}x^2$$

$$y(0)=0\rightarrow a=0\,,\quad y'(0)=0\rightarrow b=0\,,$$

$$y(L)=0\ \text{ and }\ y''(L)=0\rightarrow c=\frac{w(x)}{8EI}L^2\,, d=\frac{-5w(x)}{8EI}L$$

$$\therefore y(x)=\frac{1}{16}x^2-\frac{5}{48}x^3+\frac{1}{24}x^4$$

(c) Power series with recurrence relation 10%

Ans: Let $y=\sum_{n=0}^{\infty}c_nx^n\,,\quad y'=\sum_{n=1}^{\infty}nc_nx^{n-1}\,,\quad y''=\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}\,,\quad y'''=\sum_{n=3}^{\infty}n(n-1)(n-2)c_nx^{n-3}\,,$

$$y^{(4)}=\sum_{n=4}^{\infty}n(n-1)(n-2)(n-3)c_nx^{n-4}$$

$$\sum_{n=4}^{\infty}n(n-1)(n-2)(n-3)c_nx^{n-4}=\sum_{n=0}^{\infty}(n+4)(n+3)(n+2)(n+1)c_{n+4}x^n=\frac{w(x)}{EI}$$

$$n=0\,,\ c_4=\frac{w(x)}{24EI}\,,\quad n=1\,,\ c_5=0\,,\quad \therefore y=c_0+c_1x+c_2x^2+c_3x^3+\frac{w(x)}{24EI}x^4$$

$$y(0)=0\rightarrow c_1=0\,,\quad y'(0)=0\rightarrow c_2=0\,,$$

$$y(L)=0\ \text{ and }\ y''(L)=0\rightarrow c_3=\frac{3w(x)}{48EI}L^2\,, c_4=\frac{-5w(x)}{48EI}L$$

$$\therefore y=\frac{1}{16}x^2-\frac{5}{48}x^3+\frac{1}{24}x^4$$

2. Given differential equation as follows: $xy''+y'+xy=0$

(a) Determine the singular points of the given D.E. and classify (prove) each singular point as regular or irregular. 5%

Ans: $y''+\frac{1}{x}y'+y=0$, $x=0$ is singular point

$x\frac{1}{x}=1,\ \ x^2*1=x^2$ $\quad\therefore x=0$ is regular singular point

(b) Use the method of Frobenius to obtain the general solution. 15%

Ans: Let $y=\sum_{n=0}^{\infty}c_nx^{n+r}$, $y'=\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}$, $y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}$

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}+\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}+x\sum_{n=0}^{\infty}c_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-1}+\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}+\sum_{n=0}^{\infty}c_nx^{n+r+1}=0$$

$$\sum_{n=-1}^{\infty}(n+r+1)(n+r)c_{n+1}x^{n+r}+\sum_{n=-1}^{\infty}(n+r+1)c_{n+1}x^{n+r}+\sum_{n=1}^{\infty}c_{n-1}x^{n+r}=0$$

$$x^r\{r^2c_0x^{-1}+(r+1)^2c_1+\sum_{n=1}^{\infty}[(n+r+1)(n+r+1)c_{n+1}+c_{n-1}]x^n\}=0$$

$$\therefore r=0,0\,,\quad c_1=0 \text{ and } c_{k+1}=\frac{-c_{k-1}}{(k+r+1)^2}\quad k=1,2,3\cdots,\quad c_2=\frac{-c_0}{(r+2)^2}\,,\quad c_3=\frac{-c_1}{(r+3)^2}=0\,,$$

$$\therefore y=c_0x^r[1-\frac{1}{(r+2)^2}x^2+\cdots]=x^r[1-\frac{1}{(r+2)^2}x^2+\cdots]\,,$$

$$\frac{\partial y}{\partial r}=x^r\ln|x|[1-\frac{1}{(r+2)^2}x^2]+x^r[\frac{2}{(r+2)^3}x^2]$$

$$r=0\,,\quad \therefore y_1=1-\frac{1}{4}x^2+\cdots$$

$$r=0\,,\quad \therefore y_2=\ln|x|[1-\frac{1}{4}x^2]+\frac{1}{4}x^2+\cdots$$

$$y=\overline{c_1}y_1+\overline{c_2}y_2$$

3. (1) Consider $(x^3-2x^2+3x)^2y''+x(x-3)^2y'-(x+1)y=0$

(a) determine the singular points 5%

Ans: $x=0,3,-1$

(b) classify each singular points as regular or irregular 6%

Ans: $x=0,3\rightarrow regular$, $x=-1\rightarrow irregular$

(c) without solving the general solution, find the indicial roots about $x=0$ **6%**

Ans: $r=0,1$

4. Use the method of Frobenius to find the general solution of $x(2-x)y''-2(x-1)y'+2y=0$ **20%**

Ans: Substituting $y=\sum_{n=0}^{\infty}c_nx^{n+r}$ into the differential equation and collecting terms, we obtain

$$2(r-1+r)c_0x^{r-1}+\sum_{n=1}^{\infty}2c_n(n+r-1)(n+r)x^{n+r-1}-\sum_{n=0}^{\infty}c_n(n+r-1)(n+r)x^{n+r}$$
$$-\sum_{n=0}^{\infty}2c_n(n+r)x^{n+r}+\sum_{n=1}^{\infty}2c_n(n+r)x^{n+r-1}+\sum_{n=0}^{\infty}2c_nx^{n+r}=0$$

$$\rightarrow 4r^2c_0x^{r-1}+\sum_{k=r}^{\infty}[2c_{k-r+1}(k+1)^2-c_{k-r}(k+2)(k-1)]=0$$

$$\to 4rc_0x^{r-1}=0\,,\quad 2c_{k-r+1}(k+1)^2-c_{k-r}(k+2)(k-1)=0$$

$$\to r=0\,,\quad c_0\neq 0\,,\quad c_{k+1}=\frac{(k+2)(k-1)}{2(k+1)^2}c_k\,,\quad k=0,1,2,3,\cdots$$

$k=0\to c_1=-c_0$

$k=1\to c_2=0$

$k=2\to c_3=0$

$\vdots$

$y_1=c_0(1-x)$ set $c_0=1\to y_1=1-x$

$$y_2(x)=y_1(x)\int\frac{e^{-\int P(x)dx}}{y_1^2(x)}dx=(1-x)\int\frac{e^{-\int\frac{-2(x-1)}{x(2-x)}dx}}{(1-x)^2}dx=(1-x)[\ln x^{\frac{-1}{2}}(x-2)^{\frac{1}{2}}-1]$$

$\therefore y=C_1y_1+C_2y_2$