HOMEWORK #12 (Chapter 5 Series solutions of Linear Equations)

In problem 1 and 2, find the solution by using the Taylar series method and the recurrence relations method.

1. $y'' + x^2 y = 0$ (Exercises 5.1 Problem 14)

Ans: **Taylar series method**

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 + \cdots$$

Let $y(0) = a$, $y'(0) = b$

$y''(0) + 0 * y(0) = 0 \rightarrow y''(0) = 0$

$y''' + x^2 y' + 2xy = 0 \rightarrow y'''(0) = 0$

$y^{(4)} + x^2 y'' + 4xy' + 2y = 0 \rightarrow y^{(4)}(0) = -2a$

$y^{(5)} + x^2 y''' + 6xy'' + 6y' = 0 \rightarrow y^{(5)}(0) = -6b$

$$\therefore y(x) = a + bx - \frac{2a}{4!}x^4 - \frac{6b}{5!}x^5 + \cdots$$

recurrence relations method

Let $y = \sum_{n=0}^{\infty} c_n x^n$, $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + x^2 \sum_{n=0}^{\infty} c_n x^n = 0 \rightarrow \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2} = 0 \rightarrow$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=2}^{\infty} c_{n-2} x^n = 0 \rightarrow 2c_2 + 6c_3 x + \sum_{n=2}^{\infty} [(n+2)(n+1) c_{n+2} + c_{n-2}] x^n = 0$$

$$\therefore c_2 = 0 , \quad c_3 = 0 , \quad c_{k+2} = \frac{-c_{k-2}}{(k+2)(k+1)} \quad k = 2 \cdots \infty , \quad c_4 = \frac{-c_0}{12} , \quad c_5 = \frac{-c_1}{20} , \quad c_6 = 0 ,$$

$$c_7 = 0 , \quad c_8 = \frac{c_0}{672}$$

$$y = c_0 + c_1 x - \frac{c_0}{12}x^4 - \frac{c_1}{20}x^5 + \frac{c_0}{672}x^8 + \cdots$$

2. $y'' - 2xy' + 8y = 0, \quad y(0) = 3, \quad y'(0) = 0$ (Exercises 5.1 Problem 27)

Ans: **Taylar series method**

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 + \cdots$$

Let $y(0) = a$, $y'(0) = b$

$y''(0) + 0 * y'(0) + 8y(0) = 0 \rightarrow y''(0) = -8a$

$y''' - 2xy'' + 6y' = 0 \rightarrow y'''(0) = -6b$

$y^{(4)} - 2xy''' + 4y'' = 0 \rightarrow y^{(4)}(0) = 32a$

$$y^{(5)} - 2xy^{(4)} + 2y''' = 0 \rightarrow y^{(5)}(0) = 12b$$

$$y(0) = 3, \quad y'(0) = 0$$

$$\therefore y(x) = 3 - \frac{24}{2!}x^2 + \frac{96}{4!}x^4 + \cdots$$

recurrence relations method

Let $y = \sum_{n=0}^{\infty} c_n x^n$, $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 2x\sum_{n=1}^{\infty} n c_n x^{n-1} + 8\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} 8 c_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} 8 c_n x^n = 0$$

$$2c_2 + 8c_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} - (2n-8)c_n] x^n = 0$$

$$2c_2 + 8c_0 = 0 \rightarrow c_2 = -4c_0, \quad c_{k+2} = \frac{-(2k-8)c_n}{(k+2)(k+1)}, \quad k = 1,2,3\cdots$$

$$c_3 = c_1, \quad c_4 = \frac{c_2}{3}, \quad c_5 = \frac{c_3}{10} = \frac{c_1}{10}$$

$$y = c_0 + c_1 x - 4c_0 x^2 + c_1 x^3 - \frac{4c_0}{3} x^4 + \frac{c_1}{10} x^5 + \cdots$$